A CRITIQUE OF SOME DYNAMIC ANALYSES OF SILVER BAY, ALASKA

APPROVED:

____

____

____

____

____
Chairman

APPROVED: ______________________ DATE ____________
Dean of the College of
Mathematics, Physical
Science and Engineering

____
Vice President for Research
and Advanced Study

A CRITIQUE OF SOME DYNAMIC ANALYSES OF SILVER BAY, ALASKA

A

THESIS

Presented to the Faculty of the

University of Alaska in Partial Fulfillment

of the Requirements

for the Degree of

MASTER OF SCIENCE

By

James P. Calegari B.Sc.(Hons).

College, Alaska

18 May 1968

ABSTRACT

The equations of motion and energy for an estuarine system are set up. The mean equations are obtained and the averaging operator is investigated.

A comparison between the Silver Bay theory is made. The importance of the averaging process, the mean steady state, the tidal velocity and the lateral velocity are re-evaluated. The simplified mean equations of flow are then obtained.

The numerical equation for evaluation of pressure gradients and the velocity cross-products are obtained. Curvature effects are found to be of importance. Second order terms are investigated and numerical equations for the evaluation of v_z are found.

Some data from the Alaska Water Pollution Control Board is investigated. It is found that the Silver Bay Study neglected to consider some important factors. Lateral flows tidal motion and river run-off need more consideration.

A Statistical consideration of data is found to be important. By hypothesis testing a clearer concept of the values of the assumptions can be made. It is found that much more data is needed.

Table of Contents

Chapter 5

RECOMMENDATION FOR FURTHER STUDY

Figures

Tables

LIST OF SYMBOLS

F_i	component of force
m	mass
a	acceleration
ρ	density
$\underline{v}$	velocity vector
p	pressure
g	gravitational constant
χ	gravitational potential
μ	dynamic viscosity
∇	$\underline{i}\frac{\partial}{\partial x} + \underline{j}\frac{\partial}{\partial y} + \underline{k}\frac{\partial}{\partial z}$
$(\underline{i}, \underline{j}, \underline{k})$	unit triad vectors
η	entropy
q	net accession of heat
θ	temperature
c^2	(velocity of sound)^2
α	coefficient of thermal expansion
$\mathfrak{X}$	concentration of pollution
k_x, k_y, k_z	coefficients of diffusivity in (x,y,z) direction
S	salinity

(u, v, w)	velocity components in (x, y, z) direction
(v_x, v_y, v_z)	velocity components in (x, y, z) direction
ν	kinematical viscosity
ϵ_{ijk}	permutation tensor
V_i'	tensor velocity component
V_i	mean tensor velocity
v_i	perturbation tensor velocity
$\langle \phi \rangle = \bar{\Phi}$	time mean value of
σ_{ij}	tensor stress
δ_{ij}	kronicker delta
ϵ_{ij}	rate of strain tensor
$f = 2\omega \sin\phi$	coriolis parameter
ω	earth's angular velocity
ϕ	latitude
T	period of motion
r, θ, z	cylindrical polar co-ordinates
K_{ji}	tensor coefficients of diffusivity
$E\{X\}$	expectation value of x
$Pr.$	probability .
σ^2	variance
σ	frequency

ACKNOWLEDGMENTS

I wish to thank Dr. J. B. Matthews and Professor D. Rosenberg for their help and guidance in my graduate studies.

I am indebted to Mrs. Kathy Rhodes for aid in typing the manuscript.

This work was made possible through funds provided by the Office of Naval Research, Contract NONR 3010 (05), and by the Institute of Marine Science, University of Alaska.

CHAPTER 1

HISTORY AND MATHEMATICAL ANALYSIS OF ESTUARINE DYNAMICS

1.1 History of Estuarine Dynamics

For the last two decades attempts have been made to establish solutions to the equations of motion which would allow a description of the circulation and distribution of properties within an estuary. Most of the work has been on coastal plain type estuary. Stommel and Farmer (1952) give a review of this work. No general solution has yet been obtained, and the approximate solutions are not applicable from one estuary to another.

The circulation and structure of a fluid should be contained in the solutions to the general hydrodynamic equations.

These equations have been known for over a century. They have been discussed in detail by Stokes (1847) and Lamb (1932). Special cases have been investigated where various terms have been neglected. Proudman (1953) has given a good summary on this work. Nearly all the work prior to 1950 dealt with open ocean circulation except for the case of tidal solutions in coastal regions and estuaries (see Dronkers, 1964).

The dynamics of inlet circulation made great strides with the investigations of Stommel (1951), Lesser (1951), and Cameron (1951). First attempts were made to classify the different estuary types and to specify terms in the hydrodynamic equations which could be neglected. The ensuing simplifications led to solutions of equations which could be compared with experimental data. Poor agreement between theory and experiment could be attributed to the neglect of important terms in the original equation, lack of knowledge of the terms or to insufficient experimental data.

Stommel and Farmer (1952) reviewed the work up to 1952. They classified estuaries and compared their classification with model basin studies. In the following year Pickard (1953) produced a general description of processes in fiord estuaries. No solution to the equations of motion for estuarine environment had yet been attempted. This was due to the lack of data (see Pritchard 1954).

With the work by Maximom and Brown (1955) on the analysis of the equation of mixing, and that by Pritchard (1956) on the time mean equation of motion, came rapid advances. This was followed by the work of Pritchard and Kent (1956) on stresses in lateral flows. Stewart (1957) presented a paper showing the importance of considering the effect of curvature, and suggested a re-analysis of the Pritchard and Kent (1956) paper. Tully (1958) following the suggestion by Stewart (1957), concluded that if a section of estuary is chosen with care, the Pritchard and Kent (1956) technique can be applied.

Work around 1960 was mainly concerned with obtaining data and evaluation of terms in the equations of motion. The Silver Bay analysis by McAlister, Rattray and Barnes (1959), the study by Pickard (1961) are the classical descriptive works on fiord estuaries. Harleman and Ippen (1961) performed a study on the basic factors which determine the distribution of salinity in well-mixed and partially mixed estuaries. Bowden et al (1959) obtained experimental values of shearing stresses in tidal currents. Turbulence in tidal currents was measured by Bowden (1962), Bowden and Howe (1963) and by Grant, Stewart and Moilliet (1962). Schmitz (1962) presented a theoretical paper, discussing shearing stresses, wind stresses and velocities under different boundary conditions.

More recent work has been directed towards the evaluation of terms involved in the calculation of wind stress. Mathematical work by Rattray and Hansen (1962, 1965) has greatly simplified the equations describing flow and mixing in coastal plain estuaries. In 1966 Hansen and Rattray classified estuaries according to their relative stratification and circulation parameters. Most authors conclude that there is still insufficient theoretical and observational basis for an overall general theory.

The study by McAlister, Rattray and Barnes (1959) appears to be a fundamental paper on fiord estuary study. The work considers both the theory and experimental results in evaluating flow. The paper, however, is not very rigorous in the theoretical sense and the experimental data do not seem adequate. The study warrents a thorough analysis. This will be attempted in the pages following, but first the basic equations will be set down .

1.2 Equations of Flow.

The equations of motion are to be investigated here under certain boundary conditions. Basically the equations are those of a general Newtonian Fluid. Newton's equation of motion is the first equation to be considered:

$$\sum_{i=1}^{n} F_i = ma \qquad -(1)$$

Where F_i are the individual components of force

m is the total mass of the system

a is the acceleration of the system

Physically this equation relates the sum of the external forces acting on a system to the product of the mass and acceleration of the system. In the cases to be considered here the expression becomes the well-known Navier - Stokes equation (Rutherford 1959). The equation may be written

$$\rho \frac{d\underline{v}}{dt} = -\nabla p - g\nabla\chi + \frac{1}{3}\mu\nabla(\nabla\cdot\underline{v}) + \mu\nabla^2(\underline{v}) - 2\rho\underline{\Omega}\times\underline{v} \qquad -(2)$$

where $$\frac{d\underline{v}}{dt} = \frac{\partial\underline{v}}{\partial t} + \underline{v}\cdot\nabla\underline{v} \qquad -(3)$$

ρ = density of the fluid

$\underline{v}$ = velocity of a particle in the fluid (in vector notation)

∇ = grad operator $\nabla\cdot$ = div operator (as described by

by Sokolnifoff and Redheffer (1958)

p = pressure

g = gravitational constant

χ = gravitational potential

μ = dynamical viscosity

$\underline{\Omega}$ = Coriolis vector

The equation assumes that the motion is related to a fixed co-ordinate axis. The assumption that matter is neither created nor destroyed leads to the equation

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\underline{v}) = 0 \qquad -(4)$$

commonly called the equation of continuity. The assumption is not really valid in the general case. Atomic and nuclear theory has shown that it would be better to use a mass-energy conservation theory. But in this case, mass conservation is adequate.

One further equation, an equation of state, is required. This may be expressed by the relationship,

$$\frac{\partial \eta}{\partial t} + \underline{v} \cdot \nabla \eta = \frac{q}{\theta} \qquad -(5)$$

where η = entropy

q = net accession of heat

θ = temperature

The concept of entropy presents difficulties in applying the equation of state to the whole system. It is usual to express the equation by an equivalent form involving the velocity of sound (c) in the fluid, and the associated pressure (p). Thus the equation becomes

$$\frac{\partial p}{\partial t} + \underline{v} \cdot \nabla p - c^2 \left\{ \frac{\partial \rho}{\partial t} + \underline{v} \cdot \nabla \rho \right\} = \rho \frac{\alpha c^2}{C_p} q \qquad -(6)$$

where α = coefficient of thermal expansion.

C_p = specific heat at constant pressure.

The equation

$$\frac{\partial \mathcal{X}}{\partial t} + \underline{v}\cdot\nabla\mathcal{X} = \frac{\partial}{\partial x}\left[K_x\frac{\partial \mathcal{X}}{\partial x}\right] + \frac{\partial}{\partial y}\left[K_y\frac{\partial \mathcal{X}}{\partial y}\right] + \frac{\partial}{\partial z}\left[K_z\frac{\partial \mathcal{X}}{\partial z}\right] \quad -(7)$$

where

$\mathcal{X}$ = concentration of pollutant

K_x, K_y, K_z = coefficients of diffusivities in the (x, y, z) direction is another important equation to consider. From this equation an estimation of diffusion rates can be calculated. In particular if $\mathcal{X}$ were the concentration of salinity, equation (7) would become the salt balance equation. Rattray (1967) has expressed the salt balance equation in the form

$$\frac{\partial S}{\partial t} + \underline{v}\cdot\nabla S = -\frac{1}{b}\frac{\partial}{\partial x}\left[b\,\overline{u'S'}\right] - \frac{1}{b}\frac{\partial}{\partial z}\left[b\,\overline{w'S'}\right] \quad -(8)$$

where b = width of fjord

S = salinity

u', w', S' are deviations from the mean of velocity in x and z direction and from the salinity. The equation was obtained under the assumption that there is no lateral motion.

1.3 Expression of the Equations in Different Form.

A more useful set of equations is obtained from the general equations with certain simplifications. It is these simplifications that need to be considered carefully, that the equations can be presented in the most manageable form. The equations that need to be considered are (2) and (4).

1.31 Equation of Continuity

Recall equation (4)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{v}) = 0$$

This equation is often expressed in vector form. However it often eases algebraic manipulation if it is expressed in terms of tensors. The equation then becomes

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial v_i}{\partial x_i} + v_j \frac{\partial \rho}{\partial x_j} = 0 \qquad -(9)$$

where the subscripts are taken to denote summation according to the Einstein convention. Expressed in the expanded cartesian form, the equation becomes

$$\frac{\partial \rho}{\partial t} + \rho \left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right] + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} = 0 \qquad -(10)$$

where u, v, w are components of velocity in the (x, y, z) direction. It is interesting to note that if the Lagrangian rather than the Eulerian (formulation), had been used, equation (10) would become

$$\therefore \frac{\partial \rho}{\partial t} + \rho \left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right] = 0 \qquad -(11)$$

In the Lagrangian formulation $\frac{d}{dt} = \frac{\partial}{\partial t}$

and the co-ordinate axis is moving with the fluid.

1.32 The Navier - Stokes Equation

Basically the Navier - Stokes equation (eq.(2)) is a statement about the forces present in the system. Substituting eq. (3) in (2) results in

$$\rho\left[\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\nabla\underline{v} \right] = -\nabla p - g\nabla X + \tfrac{1}{3}\mu\nabla(\nabla\cdot\underline{v}) + \mu\nabla^2\underline{v} - 2\rho\underline{\Omega}\times\underline{v}$$

The right hand side, taken term by term represents the pressure force, gravitational force, molecular and Reynolds stresses and coriolis force. Dividing the equation by ρ and expressing it in tensor notation results in

$$\frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} - g\frac{F_i}{\rho} + \frac{\nu}{3}\frac{\partial^2 v_j}{\partial x_i \partial x_j} + \nu\frac{\partial^2 v_i}{\partial x_j \partial x_j} - 2\epsilon_{ijk}\Omega_j v_k \qquad -(12)$$

where ν = [kinematical viscosity] = μ/ρ

$$\epsilon_{ijk} = \text{permutation tensor} = \begin{cases} 1 & \text{even permute} \\ -1 & \text{odd permute} \\ 0 & \text{if any } i,j,k \text{ are equal} \end{cases}$$

Since the Navier - Stokes equation is a vector equation, it can be represented by its component equations. So, expressing it in the tensor from, it can be manipulated as a single equation, and the components can be easily obtained at any moment of the analysis.

1.4 The Time Mean Equation of Motion

In order to be able to apply numerical methods, some of the terms must be expressed in different form. Both the velocity and pressure are considered to be composed of the sum of a mean and a perturbation from the mean. This statement can be expressed by the equations

$$V_i' = V_i + v_i \qquad -(13)$$

$$P' = P + p \qquad -(14)$$

where

$$V_i = \underset{T \to \infty}{\text{Limit}} \int_0^T V_i' \, dt$$

v_i = perturbed value (mean of $v_i = 0$)

$$P = \underset{T \to \infty}{\text{Limit}} \int_0^T P' \, dt$$

p = perturbed value

V_i has the physical interpretation of being the mean flow, and P the mean pressure. Equation (12) can now be further generalized to give

$$\left[\frac{\partial}{\partial t} + (V_j + v_j)\frac{\partial}{\partial x_j}\right](V_i + v_i) = -\frac{1}{\rho}\frac{\partial (P+p)}{\partial x_i} - X_i + \frac{\nu}{3}\frac{\partial^2 (V_j + v_j)}{\partial x_i \partial x_j} + \nu \frac{\partial^2 (V_i + v_i)}{\partial x_j \partial x_j} - 2\epsilon_{ijk}\Omega_j (V_k + v_k) \qquad -(15)$$

This equation is quite general. An assumption will now be made that the fluid is incompressible. As Munn (1966) points out, this is not strictly valid. However at a depth of 100 meters, by considering the fluid to be incompressible, an error is introduced of less than 1% in the pressure term. Clearly, under these conditions, the fluid can be considered incompressible with negligible loss in generality. The equation of continuity then becomes

$$\frac{\partial \rho}{\partial t} + V_j' \frac{\partial \rho}{\partial x_j} = 0 \qquad (16\text{-a})$$

and

$$\frac{\partial V_i'}{\partial x_i} = 0 \qquad (16\text{-b})$$

Applying the operation of the mean on equation (16-b)

$$\left\langle \frac{\partial (V_i + v_i)}{\partial x_i} \right\rangle = \underset{T \to \infty}{\text{limit}} \frac{1}{T}\int_0^T \frac{\partial (V_i + v_i)}{\partial x_i}\, dt$$

and since the "x" is independent of "t" the right-hand-side becomes

$$= \frac{\partial}{\partial x_i}\left[\underset{T\to\infty}{\text{Limit}}\ \frac{1}{T}\int_0^T (V_i + v_i)\,dt\right] = \frac{\partial V_i}{\partial x_i} = 0$$

As a consequence

$$\frac{\partial V_i}{\partial x_i} = 0 \qquad -(17)$$

$$\frac{\partial v_i}{\partial x_i} = 0 \qquad -(18)$$

are obtained.

By substitution of (17) and (18) into (15) and operating with the averaging operator the Mean Navier - Stokes equation is obtained. This is

$$\frac{\partial V_i}{\partial t} + V_j\frac{\partial V_i}{\partial x_j} + \left\langle v_j\frac{\partial v_i}{\partial x_j}\right\rangle = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu\frac{\partial^2 V_i}{\partial x_j\partial x_j} - 2\epsilon_{ijk}\Omega_j V_k + X_i \qquad -(19)$$

This can be further simplified by a little algebra. If (16-b) is multiplied by $(V_i + v_i)$ it yields

$$\left\langle (V_j + v_j)\frac{\partial}{\partial x_i}(V_i + v_i)\right\rangle = 0$$

, and on expansion this becomes

$$\left\langle V_j\frac{\partial V_i}{\partial x_i}\right\rangle + \left\langle v_j\frac{\partial V_i}{\partial x_i}\right\rangle + \left\langle v_j\frac{\partial v_i}{\partial x_i}\right\rangle + \left\langle V_j\frac{\partial v_i}{\partial x_i}\right\rangle = 0$$

But $\dfrac{\partial V_i}{\partial x_i} = 0$, $\overline{v_j} = 0$ and with further algebra it can be shown that

$$\left\langle v_j \frac{\partial v_i}{\partial x_i} \right\rangle = 0$$

But

$$v_j \frac{\partial v_i}{\partial x_j} + v_j \frac{\partial v_i}{\partial x_i} = v_j \left[\frac{\partial}{\partial x_j} + \frac{\partial}{\partial x_i} \right] v_i$$

$$= \frac{\partial}{\partial x_j} (v_i v_j)$$

So

$$\left\langle v_j \frac{\partial v_i}{\partial x_j} \right\rangle = \left\langle \frac{\partial}{\partial x_j} (v_i v_j) \right\rangle \qquad (*)$$

On substitution of equation (*) into (19), the Navier - Stokes equation is obtained in the following form:

$$\frac{\partial V_i}{\partial t} + V_j \frac{\partial V_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 V_i}{\partial x_j \partial x_j} - 2\epsilon_{ijk} \Omega_j V_k - \left\langle \frac{\partial}{\partial x_j} v_i v_j \right\rangle + X_i \qquad -(20)$$

1.5 Equation of Mean Energy Flow

The energy equation can be used to determine the energy distribution within the system. It can be used to find the major energy fluctuations, and to evaluate the energy dissipated by these fluctuations.

The total stress σ_{ij} is given by

$$\sigma_{ij} = -P\delta_{ij} + \mu e_{ij} - \rho\langle v_i v_j \rangle \qquad -(21)$$

where $$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

and $$e_{ij} = \frac{1}{2}\left\{ \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right\} \qquad -(22)$$

where e_{ij} is the strain tensor, u_i is the displacement tensor, and $\rho\langle v_i v_j \rangle$ is known as the Reynolds stresses for turbulent motion. Landau and Leftshitz (1959) discuss the Reynolds stress as a measure of momentum transfer due to the turbulence. The equation of energy for the total flow can be obtained by multiplying equation (12) by V_i'. This results in

$$V_i'\frac{\partial V_i'}{\partial t} + V_i'V_j'\frac{\partial V_i'}{\partial x_j} = -\frac{1}{\rho}V_i'\frac{\partial P'}{\partial x_i} + V_i'X_i + \nu V_i'\frac{\partial^2 V_i'}{\partial x_j \partial x_j} \qquad -(23)$$

It will be noticed that the coriolis term disappears. This may be expected because the coriolis force does no real work in the system. It appears in the equation merely as a contribution due to the rotation of a co-ordinate axis. The energy equation now becomes

$$\frac{1}{2}\frac{\partial(V_i'V_i')}{\partial t} + \frac{1}{2}\frac{\partial(V_j'V_i'V_i')}{\partial x_j} = -\frac{1}{\rho}\frac{\partial(V_i'P')}{\partial x_i} + \nu V_i'\frac{\partial^2 V_i'}{\partial x_j \partial x_j} + V_i'X_i \qquad -(24)$$

Rearranging equation (24) it yields

$$\frac{1}{2}\frac{\partial (V_i')^2}{\partial t} = -\frac{\partial}{\partial x_i}\left\{V_i'\left(\frac{P}{\rho}+\frac{1}{2}V_j'V_j'\right)\right\} + \nu V_j'\frac{\partial^2 V_j'}{\partial x_i \partial x_i} + V_i'X_i \qquad -(25)$$

Using a little algebra

$$\nu V_j'\frac{\partial^2 V_j'}{\partial x_i \partial x_i} = \nu\left\{\frac{\partial}{\partial x_i}\left(V_j'\frac{\partial V_j'}{\partial x_i}\right) - \frac{\partial V_j'}{\partial x_i}\frac{\partial V_j'}{\partial x_i}\right\}$$

and the right hand side becomes

$$\nu\left\{\frac{\partial}{\partial x_i}\left(V_j'\frac{\partial V_j'}{\partial x_i}\right) + \frac{\partial}{\partial x_i}\left(V_j'\frac{\partial V_i'}{\partial x_j}\right)\right\} - \nu\left[\left(\frac{\partial V_j'}{\partial x_i}\right)^2 + \frac{\partial}{\partial x_i}\left(V_j'\frac{\partial V_i'}{\partial x_j}\right)\right]$$

From the definition of the rate of strain tensor for the total flow

$$\epsilon_{ij}' = \frac{1}{2}\left\{\frac{\partial V_j'}{\partial x_i} + \frac{\partial V_i'}{\partial x_j}\right\}$$

the energy equation finally yields

$$\frac{\partial \frac{1}{2}(V_i')^2}{\partial t} = -\frac{\partial}{\partial x_i}\left[V_i'\left(\frac{P}{\rho}+\frac{1}{2}V_j'V_j'\right)\right] + V_i'X_i + \nu\frac{\partial}{\partial x_i}\left[2V_j'\epsilon_{ij}'\right] - 2\nu\epsilon_{ij}'\frac{\partial V_j'}{\partial x_i} \qquad -(26)$$

This equation (26) states that the time rate of change in kinetic energy per unit mass is equal to (taking the terms in order from left to right) the sum of

(i) Work done by the total pressure head

(ii) Work done by the body forces

(iii) Work done by the viscous stresses

and (iv) Rate of dissipation of energy by viscosity

The mean energy flow equation is obtained by operating on equation (25) by the averaging operator. This yields

$$\frac{1}{2}\frac{\partial V_i^2}{\partial t} = -\frac{\partial}{\partial x_i}\left[V_i\left(\frac{P}{\rho}+\frac{1}{2}V_jV_j\right)\right] + \nu\frac{\partial}{\partial x_i}\left[2V_j\epsilon_{ij}\right] - 2\nu\epsilon_{ij}\frac{\partial V_j}{\partial x_i}$$
$$- V_i\frac{\partial}{\partial x_j}\left[\overline{(v_i v_j)}\right] + V_i X_i \qquad -(27)$$

The turbulent energy equation is also of interest. This can be obtained by subtracting the mean energy flow equation (27) from that of the total flow (26). The resulting equation is

$$\frac{1}{2}\frac{\partial}{\partial t}\left[\overline{v_i v_i}\right] = -\frac{\partial}{\partial x_i}\left[v_i\left(\frac{p}{\rho}+\frac{1}{2}v_jv_j\right)\right] + \left\langle \nu\frac{\partial}{\partial x_i}\left(v_j\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)\right\rangle$$
$$-\langle v_i v_j\rangle\frac{\partial V_j}{\partial x_i} - \nu\left\langle\frac{\partial v_j}{\partial x_i}\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)\right\rangle - \frac{1}{2}\frac{\partial}{\partial x_i}\left[V_i\langle v_j v_j\rangle\right] \qquad -(28)$$

Equation (27) is extremly useful, since it can be seen that the total equation need not be evaluated to obtain any meaningful results. For example, the total work done by the pressure head is the term

$$-\frac{\partial}{\partial x_i}\left[V_i\left(\frac{P}{\rho}+\frac{1}{2}V_jV_j\right)\right]$$

Expanded into cartesian form it yields

$$-\frac{\partial}{\partial x}\left[V_x\left(\frac{P}{\rho}+\frac{1}{2}V^2\right)\right]-\frac{\partial}{\partial y}\left[V_y\left(\frac{P}{\rho}+\frac{1}{2}V^2\right)\right]-\frac{\partial}{\partial z}\left[V_z\left(\frac{P}{\rho}+\frac{1}{2}V^2\right)\right] \quad (*)$$

where (V_x, V_y, V_z) are the mean velocities in the (x, y, z) direction and $V^2 = V_x{}^2 + V_y{}^2 + V_z{}^2$. The numerical evaluation of (*) can easily be carried out. The other terms of the equation can also be evaluated independent of the rest of the terms.

1.6 Equation of Motion in Cartesian Components

Since the numerical evaluation of the Navier - Stokes equation is one aim of the study, the cartesian components need to be investigated. The cartesian components can easily be obtained from the tensor equation. The transformations

$(x_1, x_2, x_3) = (x, y, z)$

$(V_1', V_2', V_3') = (U', V', W')$

$(V_1, V_2, V_3) = (U, V, W)$

$(v_1, v_2, v_3) = (u, v, w)$ or (v_x, v_y, v_z) will be used

The equations then become (x - direction)

$$\frac{\partial U'}{\partial t}+U'\frac{\partial U'}{\partial x}+V'\frac{\partial U'}{\partial y}+W'\frac{\partial U'}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+fV'+\nu\nabla^2U'+X - 2\omega\cos\phi\, W' \quad -(29)$$

y - direction

$$\frac{\partial V'}{\partial t} + U'\frac{\partial V'}{\partial x} + V'\frac{\partial V'}{\partial y} + W'\frac{\partial V'}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial y} - fU' + \nu\nabla^2 V' + Y \qquad -(30)$$

z - direction

$$\frac{\partial W'}{\partial t} + U'\frac{\partial W'}{\partial x} + V'\frac{\partial W'}{\partial y} + W'\frac{\partial W'}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z} - g - 2\omega\cos\phi\, U' + \nu\nabla^2 W' \qquad -(31)$$

The mean equations corresponding to these become

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + W\frac{\partial U}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\nabla^2 U + fV - 2\omega\cos\phi\, W + X$$
$$- \left\langle \frac{\partial}{\partial x}(v_x v_x)\right\rangle - \left\langle \frac{\partial}{\partial y}(v_x v_y)\right\rangle - \left\langle \frac{\partial}{\partial z}(v_x v_z)\right\rangle \qquad -(32)$$

$$\frac{\partial V}{\partial t} + U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y} + W\frac{\partial V}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\nabla^2 V - fU + Y - \left\langle \frac{\partial}{\partial x}(v_x v_y)\right\rangle$$
$$- \left\langle \frac{\partial}{\partial y}(v_y v_y)\right\rangle - \left\langle \frac{\partial}{\partial z}(v_y v_z)\right\rangle \qquad -(33)$$

and

$$\frac{\partial W}{\partial t} + U\frac{\partial W}{\partial x} + V\frac{\partial W}{\partial y} + W\frac{\partial W}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z} - g + \nu\nabla^2 W - 2\omega\cos\phi\, U$$
$$- \left\langle \frac{\partial}{\partial x}(v_x v_z)\right\rangle - \left\langle \frac{\partial}{\partial y}(v_y v_z)\right\rangle - \left\langle \frac{\partial}{\partial z}(v_z v_z)\right\rangle \qquad -(34)$$

1.7 The Time Averaging Operator.

The method of averaging is important to the analysis. The average is a time mean operator in the case of an estuarine system. Consider a random function $F = F(x_i, \dot{x}_i, t)$, then the time average of F is defined as

$$\langle F \rangle = \bar{F} = \underset{T \to \infty}{\text{limit}} \frac{1}{T} \int_0^T F(x_i, \dot{x}_i, t)\, dt \qquad - (35)$$

where $\dot{x}_i = \frac{\partial x_i}{\partial t}$, and T is the period of observation

Since the function F need not be a completely random function, and since there is not an unlimited amount of data, a different method of approach is needed. One possible legitimate way in which an operator could be used in the time interval (0, T), where T is finite, would be to consider the operator as a filter. That is, any periodic function can be eliminated by integrating over the period of that function, if it is of a particular form. Consider a function $f(t)$ then

$$\langle f(t) \rangle = \frac{1}{T} \int_0^T f(t)\, dt$$

Moreover if $f(t)$ is a harmonic function, say $f(t)$

$f(t) = A \cos[\sigma t + \beta]$ where A, σ, β are time independent then

$$\frac{1}{T} \int_0^T f(t)\, dt = 0$$

In particular the tidal velocity, which contributes to the overall velocity of the flow, is in the form of a set of harmonic functions. Lamb (1932) uses the tidal velocity in the form

$$U = U_o \cos(\sigma t + bx), \text{ where } \sigma, b \text{ are}$$

constants and U_o is a function of x. The period of the tidal velocity is $\frac{2\pi}{\sigma}$. If T is chosen as a multiple of $\frac{2\pi}{\sigma}$ then

$$\frac{1}{T}\int_0^{T = 2\pi n/\sigma} U \, dt = 0 \qquad -(37)$$

n = integer $\geqq 1$

If the mean square value of U is obtained, the operation yields

$$\langle U^2 \rangle = \frac{1}{T}\int_0^T U^2 \, dt = \frac{1}{2} U_o^2 \qquad -(38)$$

The method of obtaining a time mean automatically filters out all the motions which have a period of $\frac{2\pi}{\sigma n}$ (n positive integer). It can be seen that if the period is taken as that of the diurnal tide, then the only remaining effects due to the tidal motion will be in the long period terms.

It should be pointed out that if the integration is over an interval larger than the period, say nT, then the form of the operator becomes numerically evaluated. Substituting

$$U' = U + v_x + U_T$$

$$V' = V + v_y + V_T$$

$$W' = W + v_w + W_T$$

into equations (29), (30) and (31) respectively (where U_T V_T, W_T are the components of the tidal velocity), the component equations of motion are obtained. However if the coordinate axis are chosen in the correct fashion both V_T and W_T will be small enough to be neglected. The viscous stresses are only important, under normal conditions, very close to boundaries, and for most part of the flow can be neglected. Since the variation of the mean velocity over a time period is small compared to the other meaningful terms of the equation, a steady state condition can be assumed. This can be illustrated by a simple calculation. Suppose U = 10 cm./sec. and ΔU = 1 cm./sec. over a period of 1 month. Further, if ΔU = 1 cm./sec. over a distance of 1 km., then the order of $\frac{\partial U}{\partial t}$ is at least one lower than $U\frac{\partial U}{\partial x}$. However it could become very critical, for if ΔU changed by 1 cm./sec. in 10 days while ΔU change 1 cm./sec. in 2 km. then the terms would be of the same order. Under the assumptions considered, the component equations of motion become x - direction

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + W\frac{\partial U}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + fV - 2\omega\cos\phi W + X - \left\langle\frac{\partial}{\partial x}(v_x v_x)\right\rangle - \left\langle\frac{\partial}{\partial y}(v_x v_y)\right\rangle - \left\langle\frac{\partial}{\partial z}(v_x v_z)\right\rangle \qquad (40)$$

y - direction

$$U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y} + W\frac{\partial V}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial y} - fU + Y - \left\langle\frac{\partial}{\partial x}(v_x v_y)\right\rangle - \left\langle\frac{\partial}{\partial y}(v_y v_y)\right\rangle - \left\langle\frac{\partial}{\partial z}(v_y v_z)\right\rangle \qquad (41)$$

z - direction

$$U\frac{\partial W}{\partial x} + V\frac{\partial W}{\partial y} + W\frac{\partial W}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z} - 2\omega\cos\phi U - g - \left\langle\frac{\partial}{\partial x}(v_x v_z)\right\rangle - \left\langle\frac{\partial}{\partial y}(v_y v_z)\right\rangle - \left\langle\frac{\partial}{\partial z}(v_z v_z)\right\rangle \qquad (42)$$

Furthermore if the estuary is chosen such that there is little to no curvature, then V = 0 for all practicable purposes. The equations are then further simplified to

$$U\frac{\partial U}{\partial x} + W\frac{\partial U}{\partial z} + \frac{U_o}{z}\frac{\partial U_o}{\partial x} = -\frac{1}{\rho}\frac{\partial P}{\partial x} - 2\omega\cos\phi W + X - \left\langle\frac{\partial}{\partial x}(v_x v_x)\right\rangle - \left\langle\frac{\partial}{\partial y}(v_x v_y)\right\rangle - \left\langle\frac{\partial}{\partial z}(v_x v_z)\right\rangle \qquad (43)$$

$$0 = -\frac{1}{\rho}\frac{\partial P}{\partial y} + Y - fU - \left\langle\frac{\partial}{\partial x}(v_x v_y)\right\rangle - \left\langle\frac{\partial}{\partial y}(v_y v_y)\right\rangle - \left\langle\frac{\partial}{\partial z}(v_y v_z)\right\rangle \qquad (44)$$

$$U\frac{\partial W}{\partial x} + W\frac{\partial W}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z} - g - 2\omega\cos\phi\, U - \left\langle \frac{\partial}{\partial x}(v_x v_z)\right\rangle - \left\langle \frac{\partial}{\partial y}(v_y v_z)\right\rangle - \left\langle \frac{\partial}{\partial z}(v_z v_z)\right\rangle \qquad -(45)$$

Under the same assumptions that were involved in obtaining (43) (44) and (45) the mean energy equation becomes

$$\frac{\partial}{\partial t}\tfrac{1}{2}(U_0^2+U^2+W^2) = -\frac{\partial}{\partial x}\left[U\left(\frac{P}{\rho}+\frac{U^2+W^2}{2}\right)\right] - \frac{\partial}{\partial z}\left[W\left(\frac{P}{\rho}+\frac{U^2+W^2}{2}\right)\right]$$
$$+\nu\left[\frac{\partial}{\partial x}\left(2U\frac{\partial U}{\partial x}+2U_0\frac{\partial U_0}{\partial x}+U\frac{\partial W}{\partial x}+U\frac{\partial U}{\partial z}\right)\right] + \nu\frac{\partial}{\partial z}\left[2W\frac{\partial W}{\partial z}+W\frac{\partial U}{\partial z}+W\frac{\partial W}{\partial x}\right]$$
$$-2\nu\left[\left(\frac{\partial (U+W)}{\partial x}\right)^2 + 2\frac{\partial (U+W)}{\partial x}\frac{\partial (U+W)}{\partial z} + \left(\frac{\partial}{\partial z}(U+W)\right)^2\right] + UX - Wg \qquad -(46)$$
$$-U\left\{\left\langle \frac{\partial (v_x v_x)}{\partial x}\right\rangle + \left\langle \frac{\partial (v_x v_y)}{\partial y}\right\rangle + \left\langle \frac{\partial (v_x v_z)}{\partial z}\right\rangle\right\} - W\left\{\left\langle \frac{\partial (v_x v_z)}{\partial x}\right\rangle + \left\langle \frac{\partial (v_y v_z)}{\partial y}\right\rangle + \left\langle \frac{\partial (v_z v_z)}{\partial z}\right\rangle\right\}$$

1.9 General Discussion

In the preceeding sections the necessary equations for a discussion of estuaries have been presented. They will be the equations needed for a critical investigation into the Silver Bay experiment. One of the most important remarks from the previous sections was that concerning operators. The operator and its effects need to be understood so that a physical meaning can be obtained from the results. It is also important to start from the most general equation and to simplify after considering carefully the validity of the assumptions.

The component equations of motion that have been obtained, need to be investigated term wise to allow numerical evaluation. The data will then be used in the theoretical equations. From this it can be concluded that either the theory is sufficient or that it needs extensions and modifications.

CHAPTER 2

A CRITIQUE OF THE EQUATIONS USED IN THE SILVER BAY STUDY

2.0 The Critique.

A critique of any paper does not necessarily mean that one criticizes in an adverse manner. A critique is constructive rather than destructive. One reviews the assumptions, checks the validity of the argument, and at the same time tries to extend the theory into a better fitting, more rigorous one. In an experiment of the dimension of the Silver Bay study, many difficulties in data collection and physical intepretation are encountered. Technological advances overcome problems in data collection, but they do not directly aid in the development of the theory. The development of the theory is very much dependent upon thorough investigation of the assumptions involved, and a correct mathematical process in handling the raw data. In the Silver Bay study the experimental results are in some conflict with the theory used. An investigation into the assumptions and mathematical methods is necessary. In the subsequent sections of this chapter, a critique on the methodology of the Silver Bay study will be carried out.

2.1 The Longitudinal Component of the Equation of Motion:

Of the three component equations of motion, the longitudinal component is perhaps the most important. The main flow is in this direction, and the tidal effect is the greatest in this direction. Equation (29), (section 1.6) gives the longitudinal component of motion. When compared to the Silver Bay study equation (2)

$$\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + f v_y + \frac{\partial}{\partial x}(F_{x,x}) + \frac{\partial}{\partial y}(F_{x,y}) + \frac{\partial}{\partial z}(F_{x,z}) + X$$

there is a difference of $\nu \nabla^2 v_x$ and $-2\omega \cos\phi\, v_z$. McAlister, Rattray, and Barnes do not explain why they have omitted the consideration of the two terms. It was found, in the derivation of equation (40), that the terms expressing viscosity are only important in boundary considerations. They can be neglected when compared to the turbulent stresses. This is a possible explanation for the omission of the viscous term by McAlister, Rattray, and Barnes. But what are the consequences for the omission of ($-2\omega \cos\phi\, v_z$) in the equation of motion? If the term is comparable in magnitude to the other terms of the equation, then a sizable error can be introduced. A numerical consideration of this will now be shown. Consider the ratio $R = \left| \frac{f v_y}{-2\omega \cos\phi\, v_z} \right|$. Since $-2\omega \cos\phi\, v_z$ can be expressed as $-2\omega \sin\phi \frac{\cos\phi}{\sin\phi} v_z = -f \cot\phi\, v_z$, then

$$R = \left| \frac{f v_y}{-f \cot\phi \, v_z} \right| = \tan\phi \left| \frac{v_y}{v_z} \right| \qquad -(47)$$

Consider the following table:

ϕ	0°	10°	20°	30°	40°
$\tan\phi$	0	0.1763	0.3640	0.5774	0.8391
ϕ	45°	50°	60°	70°	
$\tan\phi$	1.0	1.1918	1.7321	2.7475	

If v_y and v_z are of the same order, then unless ϕ is at extreme low values or high values, the contribution of the term $-2\omega \cos\phi \, v_z$ is appreciable. The location of the Silver Bay study was at a latitude of $\sim 55^{\circ}$ N. The ratio R for this latitude becomes

$$R = 1.4281 \left| \frac{v_y}{v_z} \right| \qquad -(48)$$

If the criterion for neglecting the v_z term were that it contributed to an error of only 5%, then $|v_z| \leqq \frac{1}{14.28} |v_y|$. But it is found that $|v_z| > |v_y|$ at certain regions (see section 4.2). This clearly indicates that at least some remark should have been made by McAlister, Rattray, and Barnes about the term $-2\omega \cos\phi \, v_z$, even if later it is found to be negligible.

2.2 The Time Averaging Process.

The mathematics of the averaging process has already been discussed. The Silver Bay authors have ommitted to discuss the averaging operator. The authors do not specify the type of averaging operator, nor do they explain the physical meaning of the average. Furthermore no justification is offered for taking a period of two tidal cycles. A mention is made to the extent that the turbulent fluctuations have a time scale smaller than the period of averaging process.

The present accepted view of a mean flow cannot be defined without also specifying the time over which the average is taken. The fluctuation about the mean is called turbulence--turbulence is a statistically random process. Actually, the averaging process is a filtering precess, where turbulence with period less than two tidal periods can be filtered out from the data. This can be shown by considering the following example. Consider some turbulence to have a form of $f(t) = A \cos (bt+\gamma)$, where A, b, and γ are constants. Assume that $f(t)$ persists for a time $t \in [0, \tau]$, and $\tau < T$, the period of the averaging process. Further, assume that $\frac{2\pi}{b} < 2T$. Then it can be written that

$$\langle f(t) \rangle = \frac{1}{2T} \int_0^{2T} A \cos (bt+\gamma)\, dt$$

$$= \frac{1}{2T}\int_0 A\cos(bt+\ )\,dt$$

$$= \frac{A}{2bT}\left[\sin(b\tau+\gamma) - \sin\gamma\right] \le \frac{A}{bT}$$

If now $\tau = n\frac{2\pi}{b}$, where n is an integer, then $\langle f(t)\rangle = 0$. If $\tau > \frac{2\pi}{b}$ and $\tau \ll T$ then $\frac{A}{bT} \approx 0$ and $\langle f(t)\rangle \approx 0$. For example, if $\frac{2\pi N}{b} = T$ then $2\pi N = bT$, then $\frac{A}{bT} = \frac{A}{2\pi N}$, and for large N, $\frac{A}{2\pi N} \to 0$. It must be remembered that turbulence is a random function and will not maintain its harmonic functional form. If, however, the turbulence were present due to some persisting meteorological condition, then the time involved might be critical. If the time for which the turbulence persists is of the same order as that of the averaging process, then the mean flow will include a characteristic of the fluctuation.

2.3 The Mean Steady State:

In most cases the variation of a function with respect to time will be negligible, compared to variation with position. The flow in an estuary will depend on the quantity of fresh water entering the estuary from its various sources. When considering terms of the form $\left\langle \frac{\partial v_x}{\partial t} \right\rangle$, the time involved will be important. If the time period is that of seasonal magnitude, say of the order of three months, it would appear that $\left\langle \frac{\partial v_x}{\partial t} \right\rangle$ would be small enough to be neglected. In the Silver Bay study it was found that the March and July surface velocities were 12 cm./sec. and 18 cm./sec. respectively. This would give a value for $\left\langle \frac{\partial v_x}{\partial t} \right\rangle$ = $\frac{1}{1{\cdot}728} \times 10^{-6}$ cm./sec. In relation to the other terms of the equation it might be small. But it would not require a large physical change for $\left\langle \frac{\partial v_x}{\partial t} \right\rangle$ to become substantial enough to be considered. If the v_x was to change from 12 cm./sec. to 18 cm./sec. in a time period of one week then $\left\langle \frac{\partial v_x}{\partial t} \right\rangle \simeq$ $1{\cdot}2 \times 10^{-5}$ cm./sec. This is of the order of terms which were not neglected in the Silver Bay study. It can be concluded that it would be desirable to obtain data for the estimation of $\frac{\partial v_x}{\partial t}$. The seasonal conditions might be varied enough for $\frac{\partial v_x}{\partial t}$ term to become important.

2.4 The Tidal Velocity.

The only tidal velocity component that is usually considered is the longitudinal component. The Silver Bay study considered only this component. However Lamb (1939) has shown the importance of considering three components of tidal velocity. The components are taken to act along the co-ordinate axis. There is a definite need to find some relation between the type of estuary and the tidal components. Here it will suffice to mention only that there are three components, and not only one, as is implied by the Silver Bay study. Under certain conditions, for example strong curvature, all the components become very important. For channels, and this includes estuaries, the oscillatory tide can be expressed in the form (see Defant 1961)

$$u = \frac{ga}{c}\cos\left[\sigma\left(t-\frac{x}{c}\right)\right] - \frac{1}{8}\frac{g^2a^2}{c^3}\cos\left[2\sigma\left(t-\frac{x}{c}\right)\right] - \frac{3}{4}\cdot\frac{g^2\sigma a^2}{c^4}\,x\,\sin\left[2\sigma\left(t-\frac{x}{c}\right)\right] \quad -(49)$$

a = constant
σ = frequency of wave
c = velocity of wave

If the channel is of rectangular form, then equation (49) becomes

$$u = u_0(x)\cos\left[nt + \lambda(x)\right] \quad -(50)$$

Equation (50) is only valid if the inlet is of the rectangular type, yet it was used in the Silver Bay study. Silver

Bay has strong elements of curvature, so one would expect errors to arise in the analysis. In chapter three this will be discussed further.

The values of both $\langle U_T \rangle$ and $\langle U_T \frac{\partial U_T}{\partial x} \rangle$ have to be found where U_T is the tidal velocity $U_T = U_0 \cos\phi$. In the Silver Bay study they were obtained by averaging over two tidal cycles, and they obtained

$$\langle U_T \rangle = 0 \qquad - (51\text{-a})$$

$$\langle U_T \frac{\partial U_T}{\partial x} \rangle = U_0 \frac{\partial U_0}{\partial x} \qquad - (51\text{-b})$$

However it was found in the derivation of equation (39) and in the discussion of time average operator that equation (51-b) cannot be correct. It appears that the error has been due to the misuse of the operator. The operator that should have been used is

$$\frac{1}{nT} \int_0^{nT} \left[\qquad \right] dt$$, where n is an integer

For the case of two tidal periods the operator becomes

$$\frac{1}{2T} \int_0^{2T} \left[\qquad \right] dt$$

For a tidal velocity $U_T = U_0 \cos\phi$ one then obtains

$$\left\langle U_T \frac{\partial U_T}{\partial x} \right\rangle = \frac{1}{2T}\int_0^{2T} \frac{\partial}{\partial x}\left[\tfrac{1}{2} U_0^2 \cos^2\phi\right] dt = \frac{1}{2} U_0 \frac{\partial U_0}{\partial x}$$

It appears that the Silver Bay study used the form

$$\left\langle U_T \frac{\partial U_T}{\partial x} \right\rangle = \frac{1}{T}\int_0^{2T} \frac{\partial}{\partial x}\left[\tfrac{1}{2} U_0^2 \cos^2\phi\right] dt = U_0 \frac{\partial U_0}{\partial x}$$

A consequence of this, if the averaging period were nT, is that

$$\left\langle U_T \frac{\partial U_T}{\partial x} \right\rangle = \frac{1}{T}\int_0^{nT} \frac{\partial}{\partial x}\left[\tfrac{1}{2} U_0^2 \cos^2\phi\right] dt = \frac{n}{2} U_0 \frac{\partial U_0}{\partial x}$$

This is an obvious erroneous result. There is also a possibility that an arithmetic or printing error has been made. But whatever the reason for the error, it should be corrected. A misplaced constant often produces large errors. As an example of this, consider the differential equation $\frac{d^2x}{dt^2} = -n^2 x$ (n is constant). This has as a possible solution $x = \sin nt$. If the original equation had been $\frac{d^2x}{dt^2} = -2n^2 x$ then $x = \sin\sqrt{2}\,nt$ and in general $\sin nt \neq \sin\sqrt{2}\,nt$. In the Silver Bay study, the numerical value of $U_0 \frac{\partial U_0}{\partial x}$ was small enough to be completely neglected, when compared with other terms of the equation of motion. However, this is not always the case and when it is not the case it is imperative that the correct form is used.

A most interesting speculation can be inferred from averaging process. The tidal force over the whole inlet in one direction (on the ingoing tide), is, or should be, equal to the tidal force in the opposing direction (outgoing tide). If this were not the case there would be a net resultant tidal force. There is no non-zero net tidal force. This does not mean that there is a net zero tidal force over every section of the inlet. Quite to the contrary, in the presence of strong curvature, one would expect a net non-zero tidal force. The Silver Bay study does not consider this discussion. As will be seen later, in chapter three, it is a very important point to consider.

2.5 <u>The Lateral Velocity</u>

The lateral velocity $\tilde{v}_y$ is assumed negligible by the Silver Bay study and it was assumed

$$\tilde{v}_y = 0 \qquad -(51)$$

This is not true in general, and even when $\tilde{v}_y$ is very small with respect to $\tilde{v}_x$, it can still contribute an important amount to the equation. One needs only to consider the simple case of inertial currents (see Neumann and Pearson Jr. 1966).

Under the simplifications of a frictionless ocean, with no pressure gradients in the x and y direction, the lateral and longitudinal equations of motion become

$$\frac{dv_x}{dt} - 2\omega \sin\phi \, v_y = 0 \qquad -(52)$$

$$\frac{dv_y}{dt} + 2\omega \sin\phi \, v_x = 0 \qquad -(53)$$

This leads to the equation

$$v_y \frac{dv_x}{dt} - v_x \frac{dv_y}{dt} = 2\omega c^2 \sin\phi \qquad -(54)$$

where $c^2 = v_x^2 + v_y^2$

By allowing $\frac{u}{v} = \cot\alpha$ we can write that

$$\frac{d}{dt}(\cot\alpha) = \frac{2\omega \sin\phi}{\sin^2\alpha} \qquad -(55)$$

or that $$\frac{d\alpha}{dt} = -2\omega \sin\phi = -f \qquad -(56)$$

This equation shows that the fluid moves in a circle at a constant speed. This circle is usually called the circle of inertia. In the northern hemisphere the flow is anticyclonic while in the southern this is cyclonic. Recall that this is an effect due to the earth's rotation, the equations can be treated in a facile method if presented in cylindrical polar co-ordinates. At first sight the inertia current might be overlooked. However the physical effects of these currents are pronounced enough to be easily seen in Alaskan streams. In winter the phenomenon is apparent. For example in a stream there would exist a preferred direction for the current to flow. While the rest of the river ices over, there are open sections indicating a stronger flow. This flow is consistent with the theory of inertia currents. The preceeding discussion suggests that at any particular section, there is a non-zero mean lateral velocity. The same phenomenon is evident in Endicott Arm (S. E. Alaska), a fjord type estuary. There the movements of icebergs (surface particles) follow a path which gives a non-zero lateral component of mean velocity. The actual magnitude of the velocity could warrant further investigation. It must be remembered that there could be areas in an inlet where there will be no lateral velocity. It is interesting that the work done by the Water Pollution Control Board (1957) in Silver Bay indicates that strong lateral flows exist. Although relative to $\tilde{v}_x$ and $\tilde{v}_z$, $\tilde{v}_y = 0$ might be justified, but some data verification is required.

2.6 Equation of Mean Flow (Longitudinal Component).

It has already been pointed out that the Silver Bay study equation (2) is incorrect; the correct equation will now be obtained. If the authors of the Silver Bay study used an incorrect averaging operator, then there is a possibility that there are other errors present as a consequence of the incorrect operator.

2.61 Correlation of Perturbation and Tidal Velocity.

The statement by McAlister, Rattray, and Barnes that no relation exists between the perturbation velocities and the tidal velocity, at first sight appears legitimate. The fluctuations are considered to have periods of less than two times the tidal period. This of course allows one to consider perturbations of at least one tidal cycle. Since the fluctuations are of random nature, over a long period of time the $\langle v_i' U \rangle$ will be negligible in respect to other terms present in the equation. There may exist individual elements of $(v_i' U)$ which are sizable. As a pertinent exercise consider the following problem. Let v_x' and U have a form

$$v_x' = f(z) \exp(i\alpha(x - ct)) \qquad - (57)$$

$$U = U_0(x) \cos(\beta(x + \gamma t))$$

$$U = \frac{U_o(x)}{2}\left[\exp[i\beta(x+\gamma t)] + \exp[-i\beta(x+\gamma t)]\right] \quad -(58)$$

$(\alpha, \beta, \gamma, c)$ are constants $i^2 = -1$.

It can then be written that

$$v_x' U = \frac{U_o(x)\, f(z)}{2}\left[e^{i[\alpha x+\beta x-\alpha c t+\beta\gamma t]} + e^{i[\alpha x-\beta x-\alpha c t-\beta\gamma t]}\right].$$

and

$$\langle v_x' U\rangle = \frac{U_o(x)\, f(z)}{2T(\beta\gamma-\alpha c)}\left[\sin\left((\alpha+\beta)x+(\beta\gamma-\alpha c)T\right) - \sin\left((\alpha+\beta)x\right)\right]$$

$$- \frac{U_o(x)\, f(z)}{2T(\beta\gamma+\alpha c)}\left[\sin\left[(\alpha-\beta)x-(\beta\gamma+\alpha c)T\right] - \sin\left[(\alpha-\beta)x\right]\right]$$

On further simplification it is found that

$$\langle v_x' U\rangle \doteq U_o(x)\, f(z)\left\{\cos\left[\alpha x - \frac{\alpha c}{2}T\right]\cos\left[\beta x + \frac{\beta\gamma}{2}T\right]\right\} \quad -(59)$$

It can be seen from this result that for certain combinations of $(\alpha, \beta, \gamma, c)$ the term $|\langle v_x' U\rangle|$ can vary from a maximum of $|U_o f|$ to a minimum of 0. This means that a further investigation into the values of $U_o(x)$ and $f(z)$ is required. In the Silver Bay study it is found that $U_o(x)$ is of small enough order to be neglected. Physically $f(z)$ would also be a small term. Under these conditions the value $|U_o f|$ can be disregarded. However in other inlets this might not be the situation. Rosenberg et al (1967), found that Cook Inlet (S. E. Alaska), had large tidal components, and strong turbulence. In that case the correlation term may be very significant.

2.62 Discussion of Stress Terms

The assumption that the molecular stress terms are very much smaller than the corrosponding eddy-stress terms is justified only under certain special conditions. The molecular stress terms become very important at the boundaries. Any theory that considers the whole inlet needs to consider boundary conditions and thus molecular stresses. However in general the turbulent flow is such that the molecular agitation is trivial in comparison to the bulk motion of the eddies. Munn (1966) found that the errors in measuring the values of the eddy terms are of about the same order as the values of the molecular stresses. This is a further justification for neglecting the molecular stresses.

2.63 The Longitudinal Equation

Under all these previous considerations the Silver Bay study equation (2) is justified to the extent that two more terms need to be added. These terms are $\bar{v}_y \frac{\partial \bar{v}_x}{\partial y}$ and $-2\omega \cos\phi \bar{v}_z$.

The equation should read

$$\bar{v}_x \frac{\partial \bar{v}_x}{\partial x} + \bar{v}_z \frac{\partial \bar{v}_x}{\partial z} + \bar{v}_y \frac{\partial \bar{v}_x}{\partial y} + \frac{U_o}{2} \frac{\partial U_o}{\partial x} = -\frac{1}{\rho} \frac{\partial P}{\partial x} + f \bar{v}_y \qquad — (60)$$

$$-2\omega \cos\phi \, \bar{v}_z - \left\langle \frac{\partial (v_x' v_x')}{\partial x} \right\rangle - \left\langle \frac{\partial (v_x' v_y')}{\partial y} \right\rangle - \left\langle \frac{\partial}{\partial z} (v_x' v_z') \right\rangle + X$$

Here it is appropriate to quote from a paper by Stewart (1957). "Because of large tidal velocities and the small dimension of estuaries, it is not justified to ignore the effects of curvature of flow when computing the dynamic balance of the current system". This in effect states that even if for all practical purposes $\bar{v}_y = 0$ it is not justified to infer that $$\bar{v}_y \frac{\partial \bar{v}_i}{\partial x_j} = 0 \qquad — (61)$$

The Silver Bay study was written in 1959, two years after Stewart's paper. The most probable explanation for the results obtained by McAlister, Rattray, and Barnes is that they considered $\bar{v}_y = 0$ then $\bar{v}_y \frac{\partial \bar{v}_x}{\partial y} \simeq 0 \times \frac{\partial \bar{v}_x}{\partial y} = 0$. It would seem that the authors were unaware of Stewart's paper, or felt that it was unimportant in their case.

One reason why they may have neglected the effect of curvature is that the region investigated is rather rectangular. This is seen in figure 1 page 41. To see how important a $\bar{v}_y$ can become, and how the

FIGURE 1

LIKELYHOOD DIRECTION of the TIDAL VELOCITY

curvature of an inlet must be considered, the motion can be considered in two parts. The system can be considered as consisting of motion into two sections

(i) Inflowing Section

(ii) Outflowing Section

Consider figure 1; the flow into the inlet would follow the thick arrow, while the flow out of the inlet would follow the dotted arrow. There is no reason why the flows should follow the same paths. Furthermore, intuitively one can see a non-zero net $\bar{v}_y$. But perhaps the most interesting observation is that the tidal functions are different for ingoing and outgoing tides (later in chapter 3 this will be proved). This can be seen by considering the boundary conditions for the two sections. Let U_I be the mean ingoing tidal velocity and U_O the outgoing. The resultant mean tidal velocity over a tidal cycle would be $U_o - U_I \neq 0$. This could seem to indicate that more water is flowing out of the inlet than into, or more water is flowing in, than out of the inlet. It should actually be inferred that the flow is irregular and that in some sections $U_o - U_I > 0$ and in others $U_o - U_I < 0$ so that averaging over the whole inlet $U_o - U_I = 0$. It can also be shown the way in which the flow is affected by the curvature. Consider two sets of co-ordinates (x, y, z) and (r, θ, z)

the cartesian and cylindrical polar co-ordinates. The co-ordinate axis could be set up as in figure 2 page 44. The section that is of interest is section A. This can be considered in two parts, part I when flow is into the inlet, and part II for flow out from the inlet. The inflow conditions can be expressed by (v_x, v_y, v_z) and the outflow conditions by (v_r, v_θ, v_z). The transformation equations from one system to the other are

$$x = r \cos\theta \qquad -(61)$$

$$y = r \sin\theta \qquad -(62)$$

$$v_x = v_r \cos\theta - v_\theta \sin\theta \qquad -(63)$$

$$v_y = v_r \sin\theta + v_\theta \cos\theta \qquad -(64)$$

$$v_r = v_x \cos\theta + v_y \sin\theta \qquad -(65)$$

$$v_\theta = -v_x \sin\theta + v_y \cos\theta \qquad -(66)$$

There are many restrictions to the flow. Due to drag the currents are usually slower near the sides of a channel than they are in the middle. Kinsman (1966) suggests a rule of thumb for tidal currents, that the average velocity over a section will be about 3/4 of the central surface velocity. But for any particular θ, v_r will vary so that $v_\theta \neq 0$

then $v_y = v_r \sin\theta + v_\theta \cos\theta \neq 0$ unless $v_\theta = -\tan\theta \, v_r$,

and of course there is a possibility of this.

VELOCITY CONTOURS for EBB and FLOOD TIDES

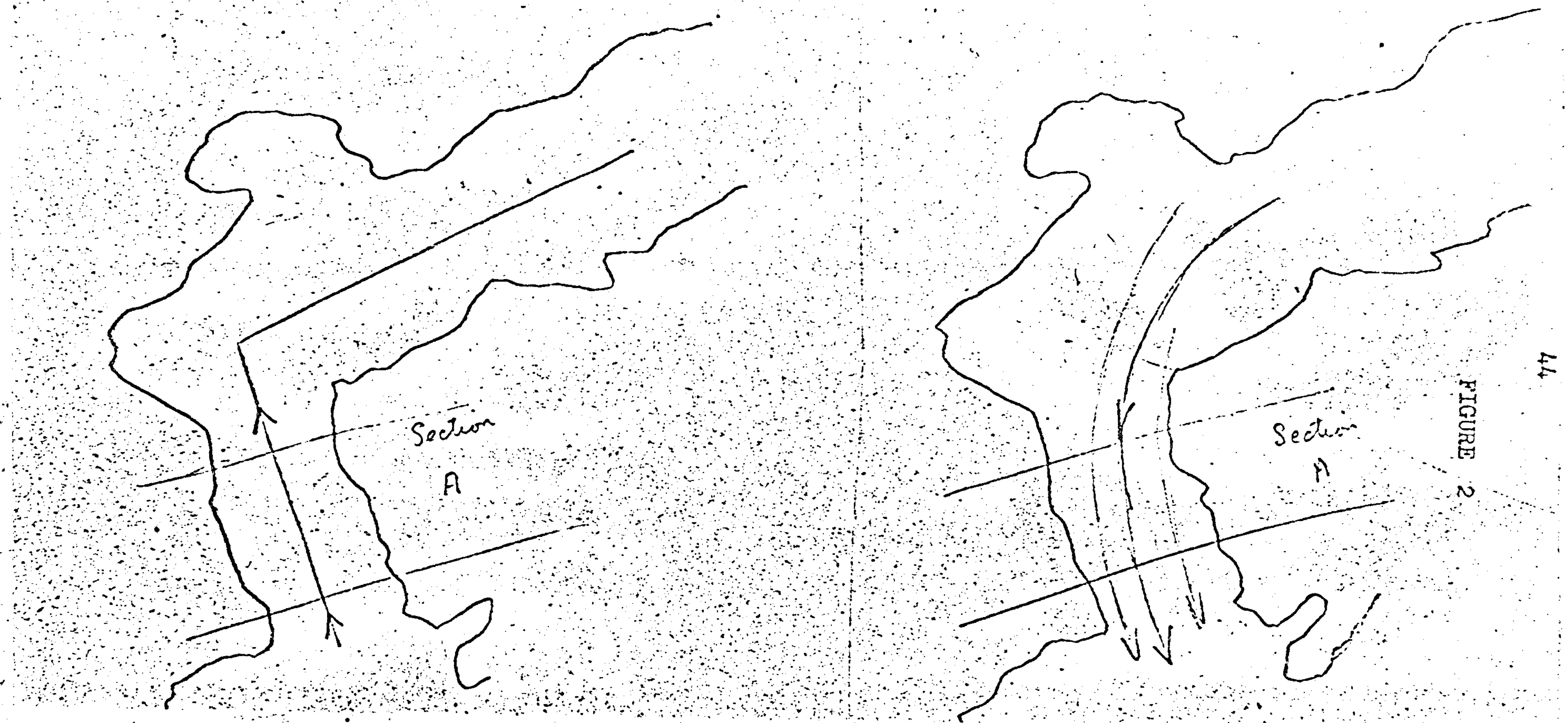


FIGURE 2

2:64 Dimensional Considerations.

In the Silver Bay study McAlister, Rattray, and Barnes considered that terms of magnitude 1×10^{-5} cm./sec.^{2} and smaller could be neglected. This allowed them to neglect the term $U_0 \frac{\partial U_0}{\partial x}$ (order $\approx 4 \times 10^{-6}$ cm./sec.^{2}). $\bar{v}_y$ was considered negligible, but how small must it be? The coriolis parameter $f \simeq 1 \times 10^{-4}$/sec. and so for $\bar{v}_y$ to be neglected the term $f\bar{v}_y < 1 \times 10^{-5}$ cm./sec.^{2} or $\bar{v}_y < 10^{-1}$ cm./sec. The Alaskan Water Pollution Studies (1957), using drift poles, computed that there existed a vector component of velocity in the y direction. In fact for the surface motion at some places $\bar{v}_y$ and $\bar{v}_x$ were of the same order.

An easy way to visualize this is by the motion in the following manner. Let V be the velocity of the motion, and let the motion be orientated θ° away from the longitudinal direction.

Then

$$\bar{v}_x = V \cos\theta$$
$$\bar{v}_y = V \sin\theta$$

and thus

$$\bar{v}_y = \bar{v}_x \tan\theta$$

At the surface of Silver Bay $\bar{v}_x \approx 10$ cm./sec., and the value of θ for $\bar{v}_y < 10^{-1}$ cm./sec. is that

$$\tan\theta < 10^{-2} \quad \text{or} \quad \theta < 36'$$

If $\bar{v}_x$ = ·1 cm./sec. then $\theta^\circ < 5^\circ 43'$. But in many places θ° is at least 20° and this suggests that more measurements taking lateral motions into consideration should be made.

Another difficulty in measuring the mean lateral flow is that the lateral fluctuations are possibly of the same order as the flow. However lateral flow does exist, and it must be considered. There is a method, for surface currents, to find this velocity. By finding a velocity path, i.e. a velocity contour, or better still the track of the maximum velocity a vector component analysis can be made. One can resolve in the x and y direction and thus find the components v_x and v_y.

2.65 Second Order Terms

The term $-2\omega \cos\phi \, \bar{v}_z$ still poses some difficulty. Two incorrect assumptions might have produced a correct result. If considering $[f\bar{v}_y - 2\omega \cos\phi \, \bar{v}_z]$, it is possible that these two parts of the expression cancel or nearly cancel each other. The error obtained by considering this assumption might be tolerable for the Silver Bay calculations. If the tables (4) and (5) in the Silver Bay study are considered as correct (in chapter 3 they will be shown to be incorrect) then the assumption is justified. This leaves

$$\bar{v}_y \frac{\partial \bar{v}_x}{\partial y} = f \bar{v}_y - 2\omega \cos\phi\, \bar{v}_z + X \qquad - (67)$$

as a second order (magnitude) equation, where X is the longitudinal component of body force. What this body force represents physically is still relatively unknown. As a first approximation it could be neglected. Since then it is possible to measure $\bar{v}_y, \bar{v}_x, \Delta y, \phi$ and ω one could numerically calculate $\bar{v}_z$. The present way that $\bar{v}_z$ is found is by considering the salt balance equations under certain simplifying assumptions. But by using equation (67) it can be shown that

$$\bar{v}_z = \left\{ \frac{f \bar{v}_y - \bar{v}_y \frac{\partial \bar{v}_x}{\partial y}}{2\omega \cos\phi} \right\} \qquad - (68)$$

This would now have to be tested to see if it is valid.

The type of analysis of equating second order terms is a very common one, in dealing with the equations of a relativistic fluid (see Chandrasekhar 1961). The method has been successful in giving physically significant solutions. Unfortunately one cannot immediately verify the equation (68) since no $\bar{v}_y$ was measured and little accurate data are known. However an estimate can be made using a possible value for $\bar{v}_y$. Now letting $\bar{v}_y = \alpha \bar{v}_x$, $\delta y = \beta \delta x$ where α and β are constants

then
$$\bar{v}_y \frac{\partial \bar{v}_x}{\partial y} \doteq \frac{\alpha}{2\beta} \frac{\partial \bar{v}_x^2}{\partial x}$$

results in
$$\bar{v}_z = \alpha \left[f \bar{v}_x - \frac{1}{2\beta} \frac{\partial \bar{v}_x^2}{\partial x} \right] \Big/ 2\omega \cos\phi$$

At the surface in Summer (July conditions)

$$\bar{v}_z = \alpha \frac{\left[10^{-4} \times 18 - \frac{1}{\beta} 15 \times 10^{-4} \right]}{1 \cdot 4281 \times 10^{-4}}$$

$$= \frac{\alpha}{\beta} \left(\frac{18\beta - 15}{1 \cdot 4281} \right)$$

If $\alpha = \frac{1}{10}$ and $\beta = \frac{3}{4}$ then

$$\bar{v}_z = \frac{4}{30} \left[\frac{-2 \cdot 5}{1 \cdot 4281} \right] \doteq -\frac{1}{4 \cdot 2} = -0 \cdot 238 \text{ cm/sec}$$

This value of $\bar{v}_z$ seems a reasonable value. One can see that with the aid of a velocity current meter where both $\bar{v}_x$ and $\bar{v}_y$ will be measured, equation (68) could give values of $\bar{v}_z$.

2.66 <u>The Body Force (longtudinal component)</u>.

The assumption X = 0 has been made; there is no reason why it should be zero. However it is possible that from

$$X = \bar{v}_y \frac{\partial \bar{v}_x}{\partial y} - f \bar{v}_y + 2\omega \cos\phi \, \bar{v}_z \qquad -(69)$$

some better understanding of a longitudinal body force can be found. By setting up a data grid where all the variables of the right hand side are measured, an analytical function can be constructed. From the constructed function it may then be possible to apply a physical interpretation.

2.67 Analysis of Results Obtained by Silver Bay Study.

It is surprising that with all the approximations and assumptions made by McAlister, Rattray, and Barnes, that good results were obtained (see Table 4 and 5 Silver Bay study). This could be attributed to the native ingenuity of the authors in making the right assumptions. This suggests that equation (68) or (69) might have meaningful physical significance. There is certainly a need for more data to evaluate the problem.

2.7 The Lateral Equation of Motion.

In considering the lateral equation, similar problems are experienced as were present in the longitudinal equation. The general equation is

$$\frac{\partial \bar{v}_y}{\partial t} + \bar{v}_x \frac{\partial \bar{v}_y}{\partial x} + \bar{v}_y \frac{\partial \bar{v}_y}{\partial y} + \bar{v}_z \frac{\partial \bar{v}_y}{\partial z} = -\left\langle \frac{1}{\rho} \frac{\partial P}{\partial y} \right\rangle - f\bar{v}_x + Y - \left\langle \frac{\partial}{\partial x}(v_x' v_y') \right\rangle - \left\langle \frac{\partial}{\partial y}(v_y' v_y') \right\rangle - \left\langle \frac{\partial}{\partial z}(v_y' v_z') \right\rangle \quad (70)$$

The question of $\bar{v}_y = 0$ is critical in equation (70). If $\bar{v}_y = 0$ equation (70) becomes

$$\left\langle \frac{1}{\rho} \frac{\partial p}{\partial y} \right\rangle + f\bar{v}_x = Y \qquad -(71)$$

Note that the Silver Bay study equation (4) is incorrect. Their equation is $-\left\langle \frac{1}{\rho} \frac{\partial p}{\partial y} \right\rangle + f\bar{v}_x = Y$

Here there is a meaning for the Y force, as that of the centripetal force due to the curvature of the channel. This is the first time that the Silver Bay authors mentions the importance of curvature, when they are considering their equation (4). When looking at equation (70) in the light of dimensions of the terms, equation (71) becomes the first order equation and

$$\bar{v}_x \frac{\partial \bar{v}_y}{\partial x} + \bar{v}_y \frac{\partial \bar{v}_y}{\partial y} + \bar{v}_z \frac{\partial \bar{v}_y}{\partial z} = 0 \qquad -(*)$$

the 2nd order equation. This gives us an alternative way of calculating $\bar{v}_z$. The equation (*) becomes

$$\bar{v}_z = -\left\{ \bar{v}_x \frac{\partial \bar{v}_y}{\partial x} + \bar{v}_y \frac{\partial \bar{v}_y}{\partial y} \right\} \Big/ \frac{\partial \bar{v}_y}{\partial z} \qquad -(72)$$

All that is needed to solve equation (72) is that $\bar{v}_x, \bar{v}_y, \Delta x, \Delta y$ be known. As in section (2.65) there are no data for $\bar{v}_y$ to be able to verify $\bar{v}_z$. But at least equation (72) warrants further investigation, for it could facilitate the evaluation of $\bar{v}_z$.

2.8 The Vertical Equation of Motion.

The equation of vertical motion is well known. It can be written as a first order

$$-\left\langle \frac{1}{\rho}\frac{\partial p}{\partial z}\right\rangle + g = 0 \qquad -(73)$$

and a second order component,

$$\bar{v}_x \frac{\partial \bar{v}_z}{\partial x} + \bar{v}_y \frac{\partial \bar{v}_z}{\partial y} + \bar{v}_z \frac{\partial \bar{v}_z}{\partial z} = 0 \qquad -(74)$$

Equation (74) could be used in checking the validity of equation (72). The $\bar{v}_y$ could be calculated from $\bar{v}_x$ and the value of $\bar{v}_z$ obtained from (72). Then a comparison of $\bar{v}_y$ (from data) and $\bar{v}_y$ (from equation (74)) could be made.

CHAPTER 3

ANALYSIS OF THE NUMERICAL METHOD

3.0 Introduction.

Theory enables one to decide what the important terms of an equation are, but methods need to be devised, so that the terms can be evaluated. In the previous chapter the important terms of the equation of motion were found. In this chapter methods of evaluating numerically each individual term will be presented. The Silver Bay study will be used as a basis for the evaluations; the Silver Bay methods will be extended to the general case where possible, and made more rigorous where necessary. It is important to point out that in a numerical evaluation of a particular term, there may be alternative equivalent ways of evaluation. The method of evaluation is dependent upon the specific conditions that the problem imposes.

3.1 The Evaluation of $\left\langle \frac{1}{\rho} \frac{\partial p}{\partial x} \right\rangle$.

In evaluating the pressure gradient, the usual method of evaluation is by use of the hydrostatic equation. The hydrostatic equation is used since it is well known, and introduces minimum error. This equation is

$$\frac{1}{\rho} \frac{\partial p}{\partial z} = g \qquad (75)$$

and on integration yields

$$p = g \int_{-\xi}^{z} \rho \, dz \qquad - (76)$$

where ξ is the displacement from mean sea level, of the surface. The equation (76) can be differentiated to yield

$$\frac{\partial p}{\partial x} = \frac{\partial}{\partial x} \left\{ g \int_{-\xi}^{z} \rho \, dz \right\} \qquad - (77)$$

Using the theorem of differentiation of integrals (see Sokolnikoff and Redheffer 1966)

$$\phi(\alpha) = \int_{u_0(\alpha)}^{u_1(\alpha)} f(x, \alpha) \, dx$$

then

$$\frac{d\phi(\alpha)}{d\alpha} = f[u_1(\alpha), \alpha] \frac{du_1(\alpha)}{d\alpha} - f[u_0(\alpha), \alpha] \frac{du_0(\alpha)}{d\alpha} + \int_{u_0(\alpha)}^{u_1(\alpha)} \frac{\partial f(x, \alpha)}{\partial \alpha} \, dx$$

Consider the pressure function, $p = p(x, y, z)$ then

$$p(x, y, z) = g \int_{-\xi(x)}^{z} \rho(x, y, z) \, dz$$

$$\frac{\partial p}{\partial x} = g\rho(x,y,z)\frac{dz}{dx} - g\rho(x,y,-\xi)\frac{d(-\xi)}{dx} + g\int_{-\xi}^{z}\frac{\partial \rho}{\partial x}dz$$

$$\frac{\partial p}{\partial x} = g\rho_s\frac{d\xi}{dx} + g\int_{-\xi}^{z}\frac{\partial \rho}{\partial x}dz \qquad -\quad (78)$$

where ρ_s is the density at the surface $z = -\xi$. This equation (78) differs from the Silver Bay study equation (8). The Silver Bay study equation (8) is

$$\frac{\partial p}{\partial x} = g\rho_s\frac{d\xi}{dx} + g\int_{0}^{z}\frac{\partial \rho}{\partial x}dz .$$

and the difference is

$$g\int_{-\xi}^{0}\frac{\partial \rho}{\partial x}dz \qquad -\quad (*)$$

The magnitude of ξ is cms, with z in terms of 100 meters. The error in neglecting the contribution of (*) is less than 0.1%, and at the same time the equation becomes simple to manage. The equation now becomes

$$\frac{\partial p}{\partial x} = g\rho_s\frac{d\xi}{dx} + g\int_{0}^{z}\frac{\partial \rho}{\partial x}dz . \qquad -(79)$$

and is valid for the degree of accuracy required. This is the same equation as the Silver Bay study equation (8). The $\frac{d\xi}{dx}$ represents the surface slope (see Neumann and Pierson 1966).

In the study of the salt balance for Silver Bay (Silver Bay study, section 5.2) the horizontal component of the turbulent flux of salt is shown to be negligible compared to the horizontal advection of salt. If it can be assumed by analogy that the horizontal component of turbulent flux of momentum is also negligible compared to the other terms in the equation of motion, then the terms

$$\left\langle \frac{\partial (v_x' v_{x_i}')}{\partial x} \right\rangle, \left\langle \frac{\partial (v_x' v_{x_i}')}{\partial y} \right\rangle$$ will drop from

the equation in which they may appear. Physically the assumption is valid, as the motion of the salt and the turbulent flux of momentum have a high correlation. The equation for longitudinal motion (equation 60) can then be written

$$\bar{v}_x \frac{\partial \bar{v}_x}{\partial x} + \bar{v}_z \frac{\partial \bar{v}_x}{\partial z} = -\left\langle \frac{1}{\rho} \frac{\partial p}{\partial x} \right\rangle - \left\langle \frac{\partial}{\partial z} (v_x' v_z') \right\rangle$$

Writing $\left\langle \frac{\partial}{\partial z} (v_x' v_z') \right\rangle = \frac{\partial}{\partial z} F(x,z)$., and using equation

equation (79) results in

$$-\rho \left\{ \bar{v}_x \frac{\partial \bar{v}_x}{\partial x} + \bar{v}_z \frac{\partial \bar{v}_x}{\partial z} \right\} = g\rho_s \frac{d\xi}{dx} + \rho \frac{\partial}{\partial z} F(x,z) + g \int_0^z \frac{\partial \rho}{\partial x} dz \quad -(80)$$

$$\frac{\partial}{\partial z} F(x,z) + \frac{g\rho_s}{\rho} \frac{d\xi}{dx} + \frac{g}{\rho} \int_0^z \frac{\partial \rho}{\partial x} dz = -\left\{ \bar{v}_x \frac{\partial \bar{v}_x}{\partial x} + \bar{v}_z \frac{\partial \bar{v}_x}{\partial z} \right\}$$

Equation (80) should correspond to the Silver Bay study equation (9)

$$\frac{\partial F(x,z)}{\partial z} + \frac{\partial \xi}{\partial x} + g \int_0^z \frac{\partial \rho}{\partial x} dz = -\rho \left\{ \bar{v}_x \frac{\partial \bar{v}_x}{\partial x} + \bar{v}_z \frac{\partial \bar{v}_x}{\partial z} \right\}$$

There are a couple of differences, but probably a result of typing errors. But more important than misprints is the fact that in equation (80), some terms have been neglected. These terms are

$$-\rho \left\{ \bar{v}_y \frac{\partial \bar{v}_x}{\partial y} - f \bar{v}_y + 2\omega \bar{v}_z \cos\phi - X \right\} \qquad (*)$$

The operations that will now be carried out on equation (80) will have to be considered carefully. The same operations on (*) might result in a breakdown of the system.

The pressure gradient term is the one being obtained. Equation (80) has to be operated upon in such a way that the term $\frac{d\xi}{dx}$ is found, since then the pressure term can be calculated from equation (79). Operating by an integral over the interval (z_1, z_2) on equation (80) results in

$$\rho\left[F(x,z)\right]_{z=z_1}^{z_2} + g\rho_s \frac{d\xi}{dx}(z_2 - z_1) + g\int_{z_1}^{z_2}\int_0^{z}\frac{\partial\rho}{\partial x}\,dz\,dz$$

$$= -\rho\int_{z_1}^{z_2}\left[\bar{v}_x\frac{\partial\bar{v}_x}{\partial x} + \bar{v}_z\frac{\partial\bar{v}_x}{\partial z}\right]dz \qquad -(81)$$

so that

$$\frac{d\xi}{dx} = -\frac{1}{\rho(z_2-z_1)}\int_{z_1}^{z_2}\int_0^{z}\frac{\partial\rho}{\partial x}\,dz\,dz - \frac{\rho}{g\rho_s}\left[\frac{F(x,z_2)-F(x,z_1)}{(z_2-z_1)}\right]$$

$$-\frac{1}{(z_2-z_1)}\frac{\rho}{g\rho_s}\left\{\int_{z_1}^{z_2}\left(\bar{v}_x\frac{\partial\bar{v}_x}{\partial x} + \bar{v}_z\frac{\partial\bar{v}_x}{\partial z}\right)dz\right\} \qquad -(82)$$

The interval (z_1, z_2) is arbitrary, so it can be chosen to simplify equation (82) as much as possible, $(z_1 \neq z_2)$. It is desirable not to choose a surface value of z, since at the surface both the inertial term and the stress terms are not well known. McAlister, Rattray, and Barnes chose the values $(z_1, z_2) = (12,100)$ meters. In this region they found that $\bar{v}_x\frac{\partial\bar{v}_x}{\partial x}$, $\bar{v}_z\frac{\partial\bar{v}_x}{\partial z}$, and $F(x,12)$ were zero, and this simplified equation (82) to

$$\frac{d\xi}{dx} = -\frac{1}{\rho(z_2-z_1)}\int_{z_1}^{z_2}\int_0^{z}\frac{\partial\rho}{\partial x}\,dz\,dz - \frac{\rho}{g\rho_s}\frac{F(x,100)}{(z_2-z_1)}$$

It can be calculated from the Silver Bay conditions that $F(x, z)$ at the bottom of Silver Bay will have a magnitude of $\approx 10^{-5}$ cm./sec.^{2} The value of $\frac{\rho}{g\rho_s} \cdot \frac{F(x,100)}{(100-12)}$ then becomes $\approx 10^{-9}$. Since the water is homogeneous, and since the lapse rate is the order of only 0.2°C/1000 meters (see Munn 1966)

the term $\int_{z_1}^{z_2} \frac{\partial \rho}{\partial x} dz = (z_2 - z_1) \frac{\partial \rho}{\partial x}$

and thus equation (82) simplifies to

$$\frac{d\xi}{dx} = -\frac{1}{\rho_s} \int_0^z \frac{\partial \rho}{\partial x} dz + \text{Terms}(10^{-9}) \qquad -(83)$$

The term that was neglected from the equation is now considered. Under the operation of the integration over (z_1, z_2) it becomes $\frac{\rho}{g\rho_s(z_2 - z_1)} \int_{z_1}^{z_2} \left(\bar{v}_y \frac{\partial \bar{v}_x}{\partial y} - f\bar{v}_y + 2\omega \bar{v}_z \cos\phi - X\right) dz$ $-(84)$

and the contribution of this is of the order of 10^{-9}. The contribution of $-\frac{1}{\rho_s} \int_0^z \frac{\partial \rho}{\partial x} dz$ is of the order of 10^{-6} so all terms of lower order can be neglected. The error introduced in neglecting the lower order terms is approximately 0.1%. Thus equation (83) becomes

$$\frac{d\xi}{dx} = -\frac{1}{\rho_s} \int_0^z \frac{\partial \rho}{\partial x} dz \qquad -(85)$$

This equation (85) is the same result as that obtained by the Silver Bay study. However the Silver Bay study does not discuss errors introduced by assuming $\left\{ \bar{v}_x \frac{\partial \bar{v}_x}{\partial x} + \bar{v}_z \frac{\partial \bar{v}_x}{\partial z} \right\} = 0$ and $F(x,12) = 0$. From equation (85) integrating over the interval $z \in (0,100)$

$$\frac{d\xi}{dx} = -\frac{1.2 \times 10^{-6}}{1.02} \simeq -1.2 \times 10^{-6} \qquad -(86)$$

It should be noticed that in the Silver Bay study (page 34) there appear to be misprints in the equations.

The analysis that has just been presented, treated the July experimental conditions. To treat the March conditions will not require any new approach. The only difference between the July and March conditions is the velocity profiles. This will mean that a different value of (z_1, z_2) needs to be considered in evaluating the $\frac{d\xi}{dx}$. It is found in the March conditions that $\frac{d\xi}{dx} = -2.1 \times 10^{-7}$ $\qquad -(87)$

The values of $\frac{d\xi}{dx}$ were sought so that a solution for equation (79) could be obtained. The numerical analysis can now be carried out to find the longitudinal pressure gradient. The Silver Bay study summarizes the analysis for the July and March conditions in tables 6 and 7. There exists an inconsistency. The results of the Silver Bay table 6 for the pressure term should be the same as the results in the Silver Bay table 4. Which table is the correct one, and why is there a difference? The results in table 4 appear to fit the theory very well. Using the raw data of salinity and temperature the terms of table 6 were recalculated. The values obtained for the pressure gradient are the same as table 6. To add further to the dilemma, the results of Silver Bay table (7) and table (5) correspond. The theory used in obtaining the results was the same as that for table (6). The reason for the difference needs to be found.

Consider table I page 61

EXTRACTS FROM THE SILVER BAY STUDY TABLE (4) AND (6)

Depth (meters)	TABLE (4) $\left\langle -\frac{1}{\rho}\frac{\partial P}{\partial x}\right\rangle$ July	TABLE (6) $\left\langle -\frac{1}{\rho}\frac{\partial P}{\partial x}\right\rangle$ July
0	-440	-120
1	-175	-70
2	-20	-25
3	-10	-11
4	-6	-6
5	-5	-6
6	-3	-5
7	-2	0
8	-1	-2
9	-0.5	0

TABLE I

From Table I it can be seen that a large discrepancy exists in only the first few meters. The basic difference between the March and July conditions is that the longitudinal surface velocity is much less in March than July. In figure 3 page 63 it can be seen that the magnitude of the July velocity for March. One could infer from this that the errors have arisen because certain terms have been neglected, the terms producing less error for smaller flows. This inference seems consistent with the treatment of the assumptions in chapter 2. It is left now to interpret this discrepancy by considering what effect the omission of the terms has produced. Recall that the terms that were neglected were

$$f\bar{v}_y, \quad \bar{v}_y \frac{\partial \bar{v}_x}{\partial y}, \quad -2\omega \cos\phi\, \bar{v}_z, \text{ and } X.$$

The terms will be studied individually to see where the error might arise. Basically the magnitude of the term

$$\left[\bar{v}_y \frac{\partial \bar{v}_x}{\partial y} - f\bar{v}_y + 2\omega \bar{v}_z \cos\phi - X \right]$$

is what needs to be investigated.

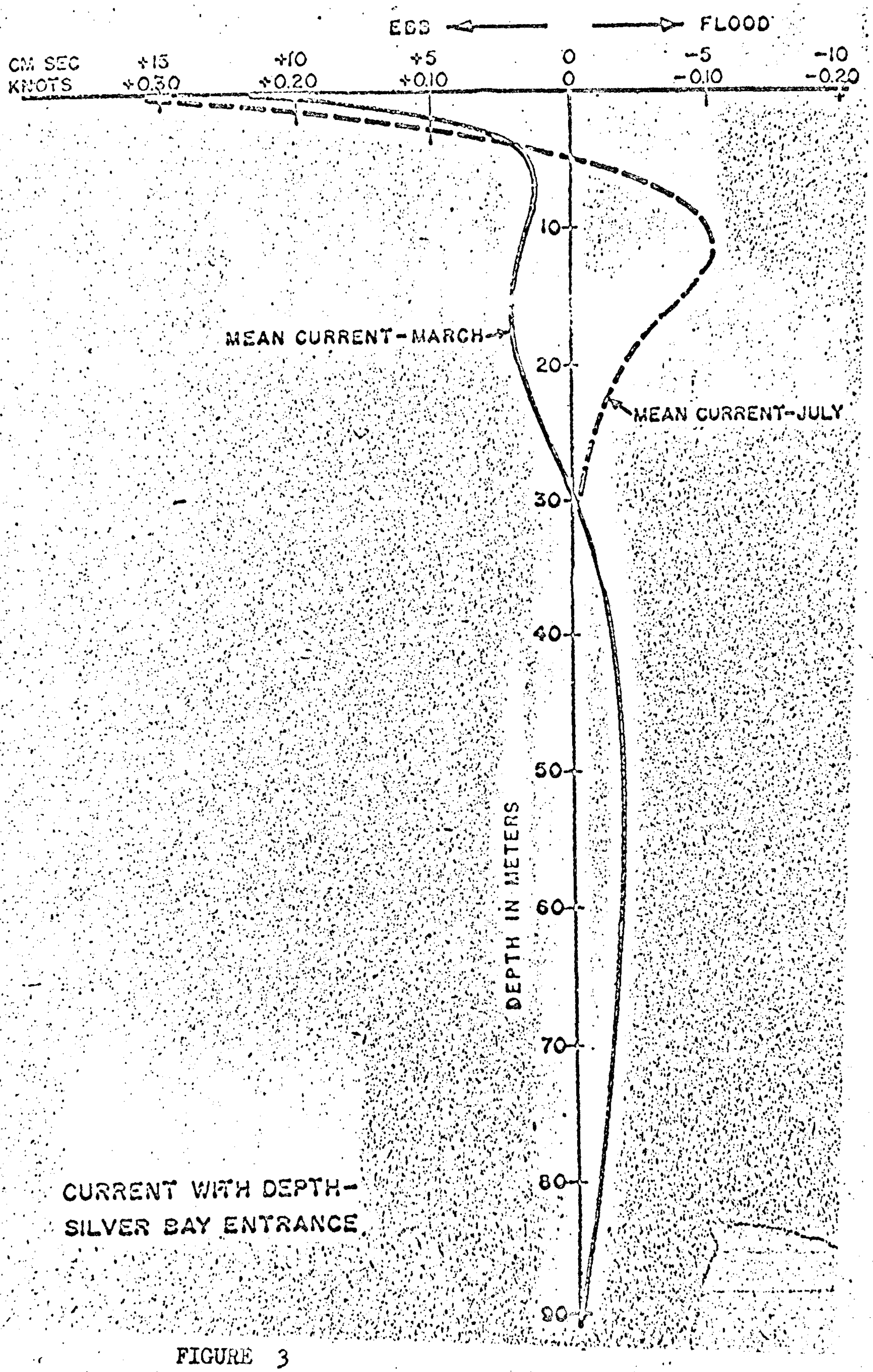

EBB
FLOOD
CM SEC
KNOTS
+15
+0.30
+10
+0.20
+5
+0.10
0
0
-5
-0.10
-10
-0.20
MEAN CURRENT-MARCH
MEAN CURRENT-JULY
DEPTH IN METERS
10
20
30
40
50
60
70
80
90
CURRENT WITH DEPTH-
SILVER BAY ENTRANCE

FIGURE 3

3.11 Effect of $\bar{v}_y \frac{\partial \bar{v}_x}{\partial y}$.

Stewart (1957) pointed out that when an inlet has a strong curvature topography, the terms $\bar{v}_y \frac{\partial \bar{v}_{x_i}}{\partial x_j}$ cannot be neglected. In particular, for the Silver Bay study the term $\bar{v}_y \frac{\partial \bar{v}_x}{\partial y}$ cannot be neglected. It is possible that if V was the surface velocity, then the component in the lateral direction could be large.

Consider the case where $\bar{v}_y = \frac{1}{10}\bar{v}_x$, and $\Delta y = \Delta x$, then $\left|\bar{v}_y \frac{\partial \bar{v}_x}{\partial y}\right| = 15 \times 10^{-5}$ cm./sec.^{2} at the surface. It is possible that $|\bar{v}_y| = |\bar{v}_x|$ and under these circumstances $\left|\bar{v}_y \frac{\partial \bar{v}_x}{\partial y}\right| = 150 \times 10^{-5}$ cm./sec.^{2}, for the surface layer. As z increases with depth, the $\bar{v}_x$ reduces rapidly, and consequently so does $\bar{v}_y \frac{\partial \bar{v}_x}{\partial y}$. Furthermore the deeper the motion, the less is the effect of wind and surface pressure. In other words there exists a damping effect with increase in depth. In the March conditions if $\bar{v}_y = \bar{v}_x$, and $\Delta y = \Delta x$ then the possible contribution at the surface of $\left|\bar{v}_y \frac{\partial \bar{v}_x}{\partial y}\right|$, would be 18×10^{-5} cm./sec.^{2} It can be seen that the summer conditions were different enough for the $\bar{v}_y \frac{\partial \bar{v}_x}{\partial y}$ term to be significant in the first few meters.

3.12 Effect of $-f\bar{v}_y + 2\omega \cos\phi \, \bar{v}_z$.

The terms ($-f\bar{v}_y + 2\omega \cos\phi \, \bar{v}_z$) have already been discussed in some detail. Here a few more comments can be made. At the surface $\bar{v}_z$ is zero or else there would be an unstable surface. Only the term $-f\bar{v}_y$ need be considered at the surface. This term can have a possible magnitude 150×10^{-5} cm./sec.^{2} if $\bar{v}_y = \bar{v}_x$, but as soon as a depth of 2 meters is reached $f\bar{v}_y \approx 1 \times 10^{-5}$ cm./sec.^{2}, neglecting $f\bar{v}_y$ would not produce large errors at depths of 2 meters. The value of $\bar{v}_z$ is of the order of only 1 cm./sec., and it would seem as if it would be unimportant at a depth of a few meters. It must be remembered that this discussion is only for the Silver Bay conditions; in another inlet where there could exist strong upwelling, and lateral currents, there would be a large contribution from the coriolis term.

3.13 Effect of the Body Force (X).

There were no body force effects (in the longitudinal direction) seen or investigated in the Silver Bay study. It is unreasonable to expect a body force to affect the motion in a different manner under the same physical conditions. If the body force were dependent on the velocity of the fluid, then the effect in March would be different than that in July. There is a definite need to investigate this longitudinal body force. More data are required so that an analytical function can be constructed, and then a physical interpretation applied.

3.14 Summary.

It seems as if the term $\bar{v}_y \frac{\partial \bar{v}_x}{\partial y}$ contributes to a large extent to the equation of motion. It is also possible that at the surface, neglecting the $f\bar{v}_y$ term, produces a sizable error, while the $-2\omega \cos\phi \, \bar{v}_z$ does not. The deficiency at the surface is -320 X 10^{-5} cm./sec.^{2} of which -300 X 10^{-5} could be accounted by the $\left\{ \bar{v}_y \frac{\partial \bar{v}_x}{\partial y} - f\bar{v}_y \right\}$. A need for more data is required.

3.2 Errors due to the Curvature.

The possibility that the errors in the Silver Bay study table (4) can be accounted by the consideration of curvature of the inlet will now be investigated. The tide has a two way motion. The effects of the tide cannot be completely filtered by taking a time average of the motion, as the tidal velocity will be slightly different in the two directions.

Consider the instantaneous velocity U_i in the positive x_i direction. Then it can be written that

$$v_i = \bar{v}_i + U_i + v_i' \qquad - (88)$$

where $\bar{v}_i$ = Eulerian mean velocity (over 2 tidal periods)

U_i = Tidal contribution to the velocity

v_i' = Turbulent contribution

It can now be shown that

$$\frac{\partial(\bar{v}_j \, \bar{v}_2)}{\partial x_i} \neq 0 \qquad - (89)$$

$$U_i \frac{\partial(U_2)}{\partial x_i} \neq 0 \qquad - (90)$$

where subscript (2) denotes lateral direction. It also follows that if the radius of curvature of the flow is R, then, in the vicinity of a data grid point, where $\bar{v}_y \approx 0$ and $U_y \approx 0$ that

$$\frac{\partial \bar{v}_y}{\partial x} = \frac{\bar{v}_x}{R} \qquad - (91)$$

$$\frac{\partial U_y}{\partial x} = \frac{U_x}{R} \qquad - (92)$$

The logical consequence of this, is if R = ∞, i.e. if the inlet has no curvature and is rectangular then by integration of (91) $\bar{v}_y$ = constant, (the constant being dependant on the boundary conditions), and U_y = constant for all x.

Note. It can be shown that if "s" is the arc length and R the radius of curvature that

$$R = \frac{\frac{dx}{ds}}{\frac{d^2y}{ds^2}} = \frac{\frac{dy}{ds}}{\frac{d^2x}{ds^2}} \quad \text{— (93)}$$

From the equation of continuity can be written that

$$\frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} = 0$$

where U_x, and U_y are the tidal component velocities. Considering figure 4, when the tide is going into the estuary, there is no curvature effect. This is equivalent to saying $U_y \approx 0$. For the outgoing tide there will be a different U_x since the tide will be affected by the shape of the inlet. Over a tidal cycle it can be written that

$$\langle U_x \rangle = \langle U_x(\text{into}) \rangle - \langle U_x(\text{out}) \rangle$$

$$= U_{o1} - U_{o2} \neq 0$$

From this, one can infer that in the longitudinal equation the terms $[U_{o1} - U_{o2}]\frac{\partial \bar{v}_x}{\partial x}$, $\bar{v}_y \frac{\partial}{\partial y}[U_{o1} - U_{o2}]$, $\bar{v}_z \frac{\partial}{\partial z}[U_{o1} - U_{o2}]$ should be included, and especially the first term could contribute an important amount to the equation. Consider $U_{o1} - U_{o2}$ = 1 cm./sec., this is reasonable since the incoming currents are often longer than the outgoing, i.e. ebb is longer than flood - (see Kinsman 1966). The value of

$$\left[U_{01} - U_{02}\right] \frac{\partial \bar{v}_x}{\partial x} \approx 15 \times 10^{-5} \text{ cm./sec.}^2 \text{ at surface}$$

$$= 9 \times 10^{-5} \text{ cm./sec.}^2 \text{ 1 meter depth}$$

$$= 3 \times 10^{-5} \text{ cm./sec.}^2 \text{ 2 meter depth}$$

When compared to other terms the above mentioned term contributes a sizable amount. It can be seen that it is unjustified to ignore the effects of curvature in the flow.

Pritchard and Kent (1956) found that the fluctuating tidal currents produced an effect of order of magnitude greater than the mean flow. It was found that it was impossible to predict the actual magnitudes of the terms involving the tidal components U_x, U_y, U_z. From the shape of the Silver Bay, there seems to be no reason to ignore the effect of curvature. There is a definite different interaction of the tide at the two sections, at the end of the data grid. At one end, there is more than a 90° change in the direction of the currents, including the tidal motion. At the other end the motion is quite rectilinear. The most probable reason why this particular grid was chosen is that the grid area is rectangular. The neglect of the discussion of the tidal effect by the Silver Bay authors is an important omission. Depending on exactly how the data has been collected, there seems to be an uncertainty in the acceptance of the Silver Bay study table (4) and (5) as being indicative statements of the mean flow. The discussion can be clarified by considering the averaging process as one which cancels opposing components in time. But this pre-requires that the components are equal and opposite. It is obvious that in the case of Silver Bay, the tidal motion is not equal and opposite.

As a summary to this section, it can be firstly stated, that more data is required. Not just information about one particular unknown, but more accurate information on all the $(\bar{v}_x, \bar{v}_y, \bar{v}_z)$ (U_x, U_y, U_z), salinity and temperature should be found. If the possibility exists, simultaneous data would be ideal. The next important conclusion, is that it is not only necessary for the grid area to be ideal (i.e. in this case rectangular), but it must also be placed in an ideal location (i.e. its boundaries must be ideal). If not, a necessary investigation into the boundary effects is required.

3.3 The Evaluation of $\bar{v}_x \dfrac{\partial \bar{v}_x}{\partial x}$.

In the evaluation of $\bar{v}_x \dfrac{\partial \bar{v}_x}{\partial x}$, the easiest and quickest way is not necessarily the best way. The assumption that

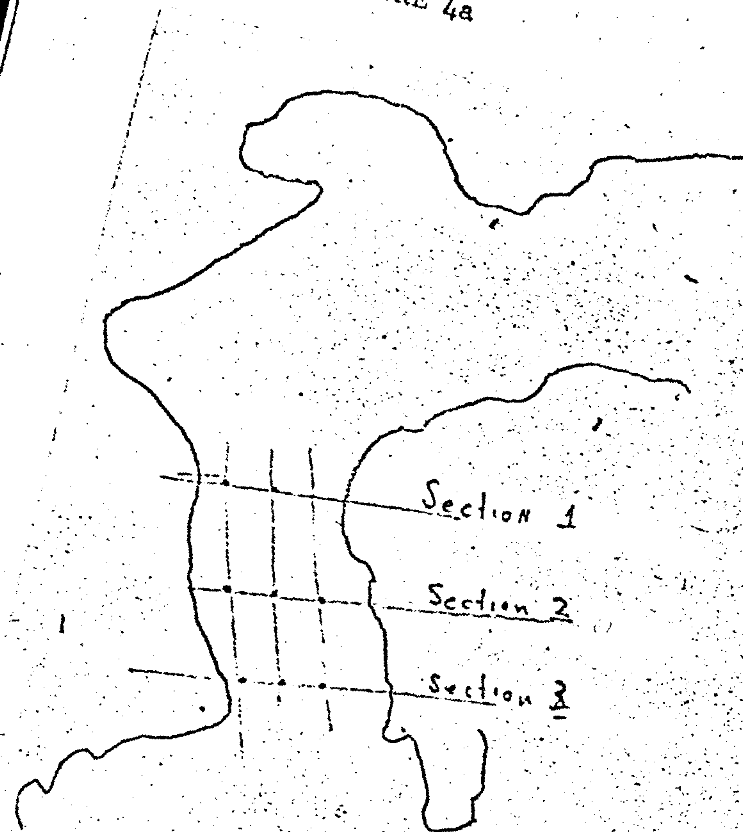
(i) Current profiles in section 1 and 3 will be similar to the measured one at section 2 (see figure 4 A)

(ii) The depth of no net horizontal motion is constant through-out the grid area,

need to be justified. Without even considering the bottom topography, the assumptions will not be valid. The tidal motion is of great importance in determining the flow, as has been shown in the previous section. The effect of the tide would place the velocity profile at any one particular grid point in doubt. One needs at least a statement of how much error is introduced in accepting the assumption. But in their own words the Silver Bay authors offer no supporting qualitative measurements. Allthough current drift appeared to be similar at the different grid points, by use of drift poles. such data are not really adequated for a serious calculation. Drift poles only give a general average picture of the mean motion.

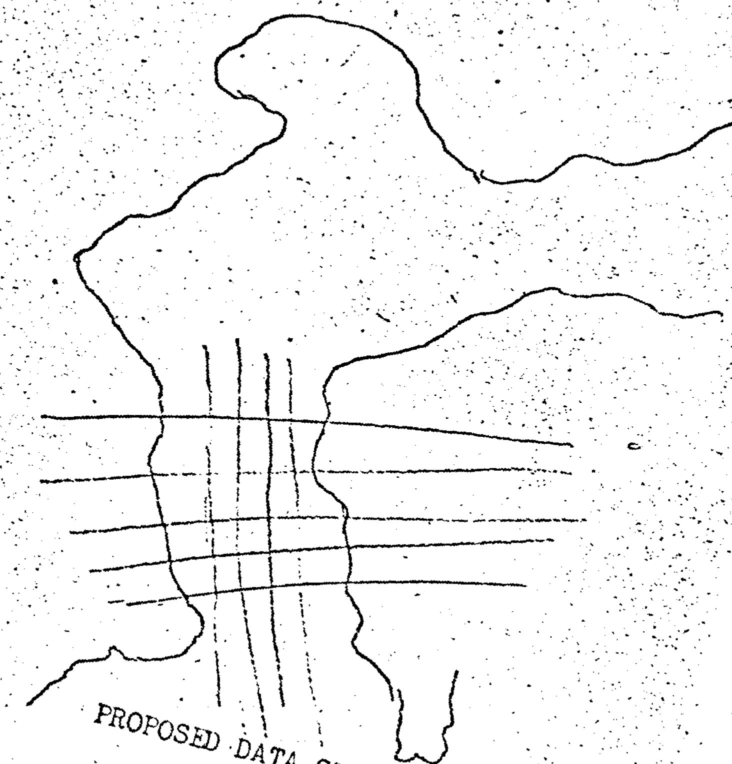
The actual numerical method of obtaining the profile at the mid-way section by the Silver Bay study method is acceptable. However it cannot be recommended toostrongly that by the aid of a set of velocity current meters at the grid points, simultaneous data can be obtained. Statistical weighting can be applied on the data, and a more representative profile would be obtained. When the data are obtained, irrespective of method, the mathematical analysis is straight

FIGURE 4a



PRESENT DATA GRID SECTION

FIGURE 4b



PROPOSED DATA GRID SECTION

forward. One considers that

$$\bar{v}_x \frac{\partial \bar{v}_x}{\partial x} = \frac{\partial}{\partial x} \tfrac{1}{2}(\bar{v}_x)^2 \qquad -(94)$$

$$\doteq \frac{1}{2} \frac{\Delta (\bar{v}_x)^2}{\Delta x}$$

and for a center grid point

$$\left.\bar{v}_x \frac{\partial \bar{v}_x}{\partial x}\right]_{x_2} \doteq \frac{1}{2}\left[\frac{\bar{v}_{x,3}^2 - \bar{v}_{x,1}^2}{\Delta x}\right] = \frac{0.69\, \bar{v}_{x,2}^2}{\Delta x} \qquad -(95)$$

However the following approach could have been used;

$$\bar{v}_x \frac{\partial \bar{v}_x}{\partial x} \doteq \bar{v}_{x,2}\left[\frac{\bar{v}_{x,3} - \bar{v}_{x,1}}{\Delta x}\right] = \frac{0.66\, \bar{v}_{x,2}^2}{\Delta x} \qquad -(96)$$

In the Silver Bay study both equation (95) and (96) are used. However, McAlister, Rattray, and Barnes use equation (95) for the layer (0,5) meters, and equation (96) from (5,100) meters. The Silver Bay authors give no reason for why they use the method described, but imply that it is a particular weighting method. It would be better to use a combination of equation (95) and (96). The combination would result in

$$\bar{v}_x \frac{\partial \bar{v}_x}{\partial x} = \frac{1}{2}\left(\bar{v}_x \frac{\partial \bar{v}_x}{\partial x}\right) + \frac{1}{2} \frac{\partial\left(\frac{1}{2}\bar{v}_x^2\right)}{\partial x}$$

$$\doteq \frac{1}{2}\frac{1}{\Delta x}\left\{\bar{v}_{x,2}\left(\bar{v}_{x,3} - \bar{v}_{x,1}\right) + \frac{\bar{v}_{x,3}^2 - \bar{v}_{x,1}^2}{2}\right\}$$

$$\bar{v}_x \frac{\partial \bar{v}_x}{\partial x}\Big]_2 = \frac{(v_{x,3} - v_{x,1})}{4 \Delta x}\left(\bar{v}_{x,3} + 2\,\bar{v}_{x,2} + \bar{v}_{x,1}\right) \quad - (97)$$

On inspection of the Silver Bay study table 9, and results obtained by equation (97), the results differ by only 1%. However the equation (97) is statistically more sound than either (95) or (96) as it considers the profile as one entity, rather than two separate sections. The difference in the results, however, indicate that one method is not much more superior than the other.

The need to take a finer grid, than the one used by the Silver Bay study is necessary. With a finer grid more accurate data can be obtained, this in turn gives rise to better solutions. By inspecting figure (4 b), one can see that a 4 X 5 grid area could be used. The finer grid would give more data for a more accurate $\bar{v}_x \frac{\partial \bar{v}_x}{\partial x}$. The actual size of the grid is dependant very much on the type of order and accuracy of the variable under investigation. Obviously, it would be ridiculous to take grid size of 1 cm., when the error in defining position is of the order of 100 meters. Hence, if one wanted at least a 90% accuracy, a grid size of 1 km. or more would be needed, if error in positioning were 100 meters.

Another reason for needing a finer grid area, is that more profile sections would be obtained. With three sections, only one mean profile can be obtained. With four sections, say (S1, S2, S3, S4) then $\overline{S2}$, would be mean of (S1, S3), $\overline{(S3)}$ would be mean of (S2, S4). Further (S1, S4) would give a mean between S2, and S3 say $\overline{SA}$ and (($\overline{S2}$), ($\overline{3}$)) would give $\overline{\overline{SA}}$ then $\overline{SA}$ and $\overline{\overline{SA}}$ would give a final mean profile. Obviously, 4 sections are superior to three sections. With 5 sections, (S1, S2, S3, S4, S5), an $\overline{S2}$, $\overline{S3}$, $\overline{S4}$ would result from (S1, S3), (S2, S4), (S3, S5). Further ($\overline{S2}$, $\overline{S4}$) would give $\overline{\overline{S3}}$. When there is more than one mean profile, the amount of variation existing would be clear from the differences in the two mean profiles. One would strongly suggest, that at least two mean profile sections be obtained.

In the evaluation of the March data, there are the further problems, that the velocities are smaller and the inaccuracies in the measurements are increased. Also one wonders if the average of 20 and 17 is 18, (Silver Bay study, table (10) first entry).

3.4 The Evaluation of $\bar{v}_z \frac{\partial \bar{v}_x}{\partial z}$.

In the evaluation of the term $\bar{v}_z \frac{\partial \bar{v}_x}{\partial z}$, the method described by the Silver Bay study is the usual way it is evaluated. The $\mathcal{V}_{z}$ is computed by the continuity equation; by calculating the salt transport over a particular surface z = h, the at the surface h is found. The equation for the diffusion of salt in water, in an estuarine environment is known.

Only special solutions can be obtained analytically, and this usually requires assumptions which linearizes the general equation. As a consequence one feels uneasy in accepting a calculated from a salt balance equation. It would be ideal if some analytical method could be developed. Some experimental technique could be developed; a special type of Eckman meter based upon vertical fluctuations seems to be a possibility. A more sophisticated method would use an electronic current meter, which could be placed on its side, and instead of registering horizontal currents it would give vertical fluctuations.

There are technical problems associated with measurements of currents, but at least some attempts should be made. The smallness in the magnitude of would present problems of sensitivity for the current meter. The present current meters used by the Institute of Marine Science (University of Alaska) can register down to 2.5 cm./sec. However a meaningful could be 0.001 cm./sec. One possible alternative, as mentioned in the last Chapter, is to employ a second order equation of motion to find a $\bar{v}_z$, from measured values of $\bar{v}_y$ and $\bar{v}_x$.

3.5 The Evaluation of $\left\langle \frac{1}{\rho} \frac{\partial P}{\partial y} \right\rangle$.

The terms needed to evaluate the longitudinal equation have been considered, it is now important to evaluate terms affecting the dynamics in the transverse direction. The Silver Bay authors have no data for the transverse direction, and no real check can be made to see if the theory is satisfied. One can at least develope the equations needed for the numerical analysis.

It was found that equation (71) is the important equation describing the dynamics in the transverse direction. Basically the equation is a first order equation plus a second order equation. The first order equation is

$$\left\langle \frac{1}{\rho} \frac{\partial P}{\partial y} \right\rangle + f \bar{v}_x = Y$$

and the second order is

$$\bar{v}_x \frac{\partial \bar{v}_y}{\partial x} + \bar{v}_y \frac{\partial \bar{v}_y}{\partial y} + \bar{v}_z \frac{\partial \bar{v}_y}{\partial z} = 0$$

Using the same methodology as in section 3.1

$$p = g\int_{-\xi}^{z} \rho \, dz$$

$$\frac{\partial p}{\partial y} = g\rho_s \frac{d\xi}{dy} + g\int_{-\xi}^{z} \frac{\partial \rho}{\partial y} dz \qquad -(98)$$

one obtains

$$g\frac{\rho_s}{\rho}\frac{d\xi}{dy} + \frac{g}{\rho}\int_{-\xi}^{z} \frac{\partial \rho}{\partial y} dz + f\bar{v}_x - Y = 0 \qquad -(99)$$

Note that the Silver Bay study equation (21) is

$$-\left\langle \frac{1}{\rho}\frac{\partial p}{\partial y} \right\rangle + f\bar{v}_x = Y$$

and this is an incorrect equation (see Neumann and Pierson 1966). Without repeating section 3.1, if appropriate (Z_1, Z_2) are taken for the first integration, one obtains the result

$$\frac{d\xi}{dy} = -\frac{1}{\rho_s}\int_{0}^{z} \frac{\partial \rho}{\partial y} dz \qquad -(100)$$

This equation has the same form as (85), and this is expected, since the same method is used. The term $(\frac{d\xi}{dy})$ can now be evaluated from equation (100), substituted into equation (98) where the pressure gradient can be found.

3.6 The Centripetal Force.

The centripetal force can be calculated indirectly from equation (71), or directly by examining the radial velocity. Indirectly, from knowing the pressure gradient and the longitudinal velocity, the value of Y can be found. Since the radial force per unit mass is $\frac{v_r^2}{R}$ where R is radius of curvature of flow, and expressing v_r^2 in terms of $\bar{v}_x$, and $\bar{v}_y$ as suggested in equations (63) - (66), Y can be found. It would then be left to check the equation (71) for validity.

3.7 Some Considerations on the Second Order Terms.

As mentioned in chapter 2, second order equations sometimes give physically significant results. Consider the equation

$$\bar{v}_x \frac{\partial \bar{v}_y}{\partial x} + \bar{v}_y \frac{\partial \bar{v}_y}{\partial y} + \bar{v}_z \frac{\partial \bar{v}_y}{\partial z} = 0$$

then over the sections (1, 2, 3), it can be written as

$$\bar{v}_{x,2}\left[\frac{\bar{v}_{y,1} - \bar{v}_{y,3}}{\Delta x}\right] + \bar{v}_{y,2}\left[\frac{\bar{v}_{y,1} - \bar{v}_{y,3}}{\Delta y}\right] + \bar{v}_{z,2}\left[\frac{\bar{v}_{y,1} - \bar{v}_{y,3}}{\Delta z}\right]$$

Since the grid section is unchanging, if set up correctly then $\Delta x : \Delta y : \Delta z :: \alpha_1 : \alpha_2 : \alpha_3$ where ($\alpha_1, \alpha_2, \alpha_3$) are constants. Then one can obtain

$$\left(\bar{v}_{y,1} - \bar{v}_{y,3}\right)\left\{ \frac{\alpha_2 \alpha_3 \bar{v}_{x,2} + \alpha_1 \alpha_3 \bar{v}_{y,2} + \alpha_1 \alpha_2 \bar{v}_{z,2}}{\alpha_1 \alpha_2 \alpha_3} \right\} = 0$$

or that

$$v_{z,2} = -\left\{ \frac{\alpha_2 \alpha_3 \bar{v}_{x,2} + \alpha_1 \alpha_3 \bar{v}_{y,2}}{\alpha_1 \alpha_2} \right\} \qquad - (101)$$

To give an idea of what type of results can be obtained from (101), a few examples will be given.

CASE I: Assume $\alpha_1 = \alpha_2 = 10^3$ meters, $\alpha_3 = 1$ meter, $\bar{v}_{x,2} = 10$ cm./sec. and $\bar{v}_{y,2} = 0$ then $\bar{v}_{z,2} = -0.01$ cm./sec.

CASE II: Assume same conditions as in case 1, except that $\bar{v}_{y,2} = 1$ cm./sec. then $\bar{v}_{z,2} = -0.011$ cm./sec.

CASE III: Assume conditions as in case 1 except that $\bar{v}_{x,2} = 1$ cm./sec., then $\bar{v}_{z,2} = -0.001$ cm./sec.

Comparing the results that the Silver Bay study obtained for the $\bar{v}_z$ it appears that the order of $\bar{v}_z$ obtained in case I, II and III are of the correct magnitude. This is encouraging since equation (101) is straightforward to evaluate. A further excercise will now be done, and a comparison will be made between the results given by equation (101) and the Silver Bay study table (11). Consider $\alpha_1 = 1500$ meters, $\alpha_3 = 1$ meter, $\alpha_2 = 750$ meters then

$$\bar{v}_{z,2} = -\left\{ \frac{750\,\bar{v}_{x,2} + 1500\,\bar{v}_{y,2}}{1.125 \times 10^6} \right\} cm/sec$$

(assume $\bar{v}_y = 0$)

Comparison of the $\bar{v}_z$

z (meters)	$\bar{v}_{x,z}$ (cm/sec)	$\bar{v}_{z,z}$ calculated	$\bar{v}_{z,z}$ Table (Silver Bay)
0	18		
		-0.012	-0.005
1	14		
		-0.009	-0.010
2	9		
		-0.006	-0.014
3	6		
		-0.004	-0.016
4	3		
		-0.001	-0.018
5	1		
		-0.0006	-0.017
6	-0.5		
		+0.0003	-0.017
7	-2		
		+0.001	-0.017
8	-3		
		+0.002	-0.017
9	-4		
		+0.0024	-0.014
10	-5		

TABLE II

It is interesting to note the difference that exists in the values given by the two systems. Immediately it can be seen that $\bar{v}_y \neq 0$ would result in a different $\bar{v}_z$, but what is $\bar{v}_y$? The most interesting difference is the change in sign at the 6 meter level. Since this region is also a region of zero longitudinal velocity, one may expect a stagnation region, and if so a reversal in sign of $\bar{v}_z$ would be expected.

Equation (74) is another second order equation. Using the same $(\alpha_1\ \alpha_2\ \alpha_3)$ as described above, equation (74) becomes

$$\frac{(\bar{v}_{z,1} - \bar{v}_{z,3})}{\alpha_1 \alpha_2 \alpha_3} \left\{ \bar{v}_{x,2}\, \alpha_2 \alpha_3 + \bar{v}_{y,2}\, \alpha_1 \alpha_3 + \bar{v}_{z,2}\, \alpha_1 \alpha_2 \right\} = 0 \qquad -(102)$$

On simplifying one obtains

$$\bar{v}_{z,2} = - \left\{ \frac{\alpha_2 \alpha_3\, \bar{v}_{x,2} + \bar{v}_{y,2}\, \alpha_1 \alpha_3}{\alpha_1 \alpha_2} \right\}$$

which, interestingly, is the same as equation (101). It seems then, that the second order terms of both the lateral and vertical motion give rise to the same form for $\bar{v}_z$. This is perhaps a strong indication that the equation (101) does have some valuable physical significance. There is definitely a need for further investigation.

3.8 <u>Salt Balance Equation</u>.

Only a brief comment will be given on the salt balance equation. The salt transport allows for the calculation of the $\tilde{v}_z$, so the equation is important. As was shown in chapter I, the general equation can be written as

$$\frac{\partial S}{\partial t} + v_j \frac{\partial S}{\partial x_j} = \frac{\partial}{\partial x_j}\left(K_{ji} \frac{\partial S}{\partial x_j}\right) \qquad (*)$$

where K_{ij} are the diffusivities, and $v_j \frac{\partial S}{\partial x_j}$ being the advection of salt. Using the continuity equation $\frac{\partial v_i}{\partial x_i} = 0$, (*) then becomes

$$\frac{\partial S}{\partial t} = - \frac{\partial (v_j S)}{\partial x_j} + \frac{\partial}{\partial x_j}\left(K_{ji} \frac{\partial S}{\partial x_j}\right) \qquad -(103)$$

If one considers the main flow area, one can neglect the diffusivity terms, and then the equation becomes simple to evaluate. The diffusivity terms become important near the bottom and sides of the estuary.

3.9 Summary.

In this chapter the numerical methods of evaluating the dynamics of an estuary were investigated. It was found curvature affected the flow, especially the tidal motion. A finer data grid is found to be desirable. Simultaneous data would be extremely useful. More than just a $\tilde{v}_x$ is important in the analysis, and $\tilde{v}_y$ data are imperative to obtain. The possibility of evaluating $\bar{v}_z$ from $\bar{v}_x$ and $\bar{v}_y$ was investigated and showed promise, more data are required.

CHAPTER 4

ANALYSIS OF THE CONCLUSIONS OF THE SILVER BAY STUDY

4.0 Introduction.

The validity of the conclusions is dependant on the validity of the assumptions involved in obtaining the conclusions. In chapter I a general mathematical analysis was presented. The theory of the Silver Bay analysis was discussed in chapter II, and in chapter III the numerical methods were evaluated. It is now left to evaluate the conclusions obtained by the Silver Bay authors. As has been stated, some of the assumptions were invalid. Therefore erroneous results are expected. The results will now be reviewed in the light of other data, and an alternative theory.

4.1 General Discussion of Some Physical Observations.

The results of the Silver Bay study were obtained under some assumptions which may not hold for Silver Bay. This does not mean that the results are erroneous, but rather that the confidence level of acceptance is somewhat reduced. The confidence level is defined as the % probability, i.e. if the confidence level of accepting A is 95% then the probability that A is correct is 0.95, (see Fraser 1961).

An example of a non-valid assumption is that the lateral velocity is zero. McAlister, Rattray, and Barnes assume that the lateral velocity is zero, and make no attempt to measure it. In their own words "No attempt was made to calculate the lateral velocities..." The measuring of a longitudinal and lateral velocity has, since the time of the Silver bay study, become quite sophisticated. Lateral velocities can be measured to a high order of accuracy, $\pm$ 3% error in the range 0.05 to 7.0 knots (see Hydro Products 1967). This means that a lateral velocity of 2 cm./sec. at the surface can be measured. In the Silver Bay study Ekman, Biplane, and Magnesyn meters were used. These meters, although they can measure a wide range of velocity, have a low order of accuracy, $\pm$ (15 - 25%) error. This is due to the mechanical make-up of the meters. William von Arx (1964) discusses these meters in detail. The use of meters, since they register a distribution of velocity, and since the Silver Bay grid area has a complex topography, give rise to errors in the interpretation. One can only infer the direction and the approximate magnitude of the current, in the direction of the mainflow. This is a definite limitation.

Whenever assumptions are made, or results are given, a confidence level should also be given.

This confidence level would allow one to interpret how valid the results could be. The variance from the mean is an important quantity, since besides giving an idea of the size of the perturbations, confidence levels can be obtained. An example of this will now be given. Consider the term $\bar{v}_x \frac{\partial \bar{v}_x}{\partial x}$ with a mean m and variance σ^2. Then if one rejects terms which have a 30% error, one would reject $\bar{v}_x \frac{\partial \bar{v}_x}{\partial x}$ if

$$Pr. \left\{ -\alpha \leq \frac{m - \mu}{\frac{\sigma}{\sqrt{n}}} \leq +\alpha \right\} \leq 0.70$$

n = data elements

Pr = probability, μ = theoretical mean, α = constant

The term $v_x \frac{\partial \bar{v}_x}{\partial x}$ would have a $\frac{2\sigma}{m}$ X 100% error range. Giving a result with a confidence level, enables one to decide if more data are needed or if ammendments have to be made to the theory.

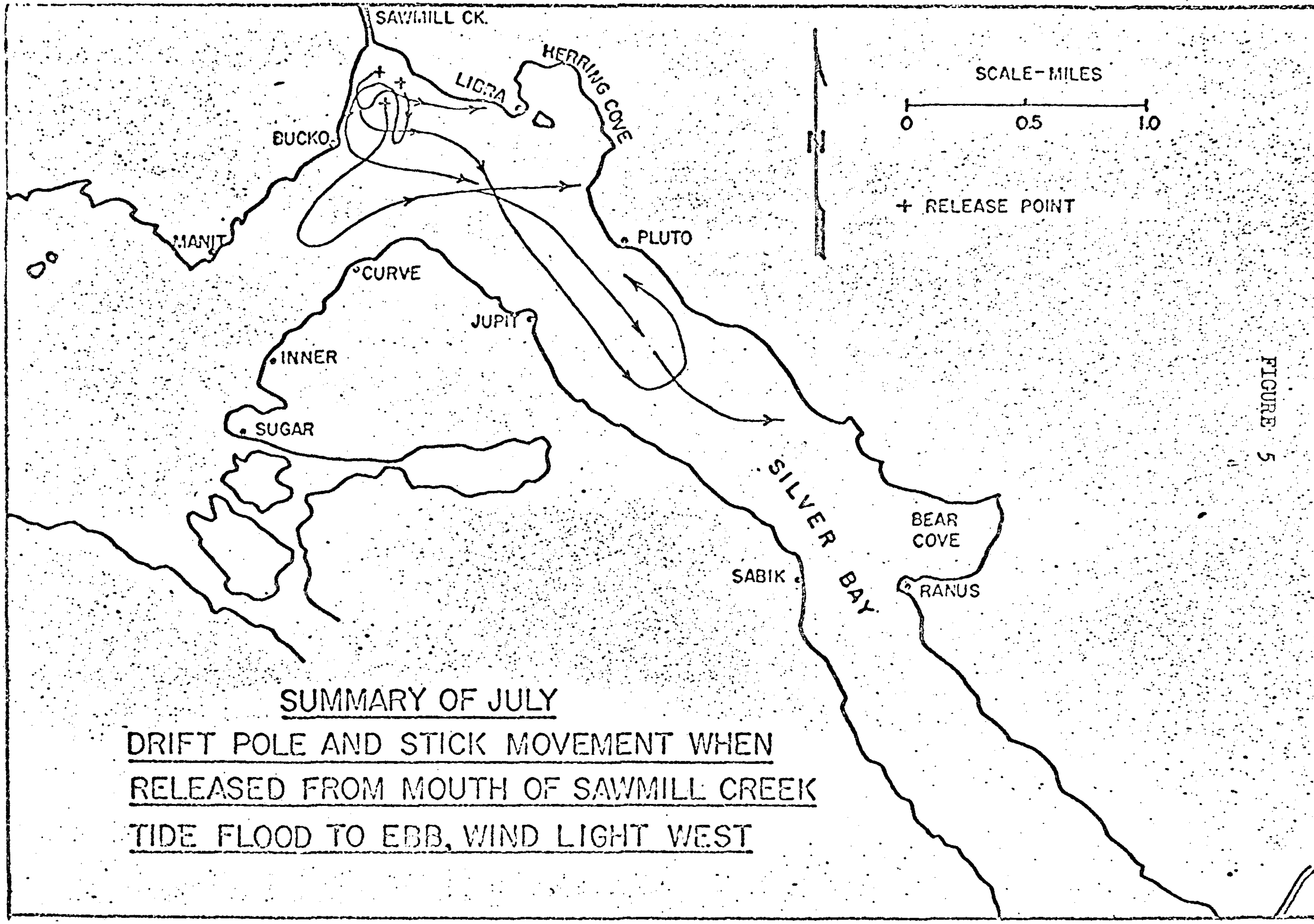
4.11 The Alaska Water Pollution Control Board.

The Alaska Water Pollution control board carried out a set of pollution control studies in Silver Bay in 1956 - 57. They published their findings in Water Pollution Control (1957). The region of study coincided with the Silver Bay study area. Although the Alaska Water Pollution Control

Board had a different aim in mind than that of the Silver Bay authors, current studies were made. Pertinent information was published by the Alaska Water Pollution Control Board, which is consistent with the theory presented in chapter 2 and 3 and contradictory to the Silver Bay analysis. It was found that the meteorological conditions existing during the time of the data collection was severe enough to affect the flow. For example, the strong ~5-10 knots westerly winds reversed the surface currents for a day or so. No mention of this is given in the Silver Bay study. The collecting of simultaneous meteorological and oceanographic data is essential. If one has the required data, by carefully filtering, one can correct the bias. In the next few sections the relevant parts from the Alaska Water Pollution Control Board study will be discussed. In the light of the Alaska Water Pollution Control Board study the conclusions of the Silver Bay study will then be discussed.

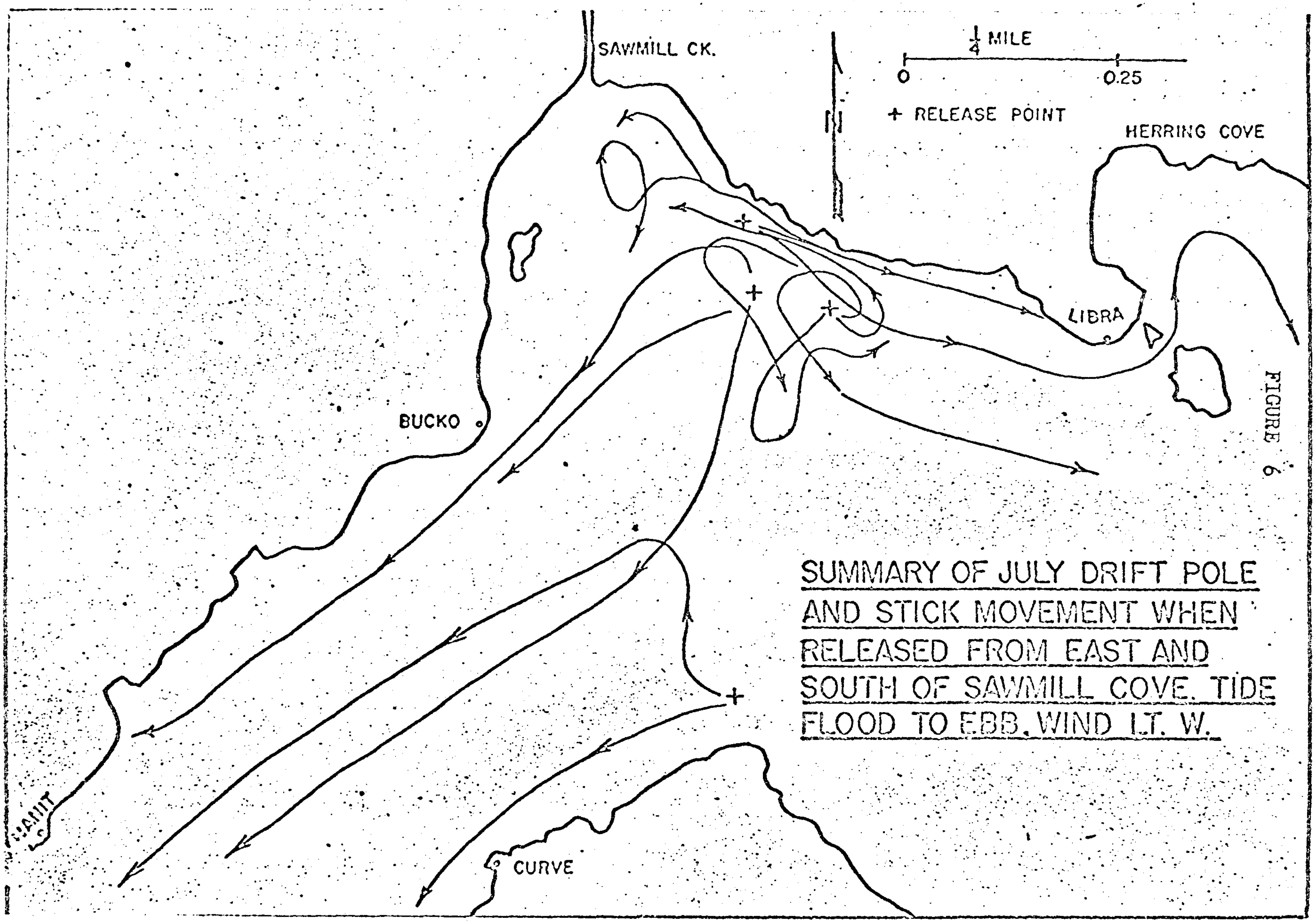
4.2 Evidence of Lateral Flows.

The Alaska Water Pollution Control Board gives a diagramatic summary of flow for the July condition, and from these diagrams the existence of lateral flows can be inferred. The summary is presented in figure (5) and (6). From



SUMMARY OF JULY
DRIFT POLE AND STICK MOVEMENT WHEN RELEASED FROM MOUTH OF SAWMILL CREEK
TIDE FLOOD TO EBB, WIND LIGHT WEST

FIGURE 5



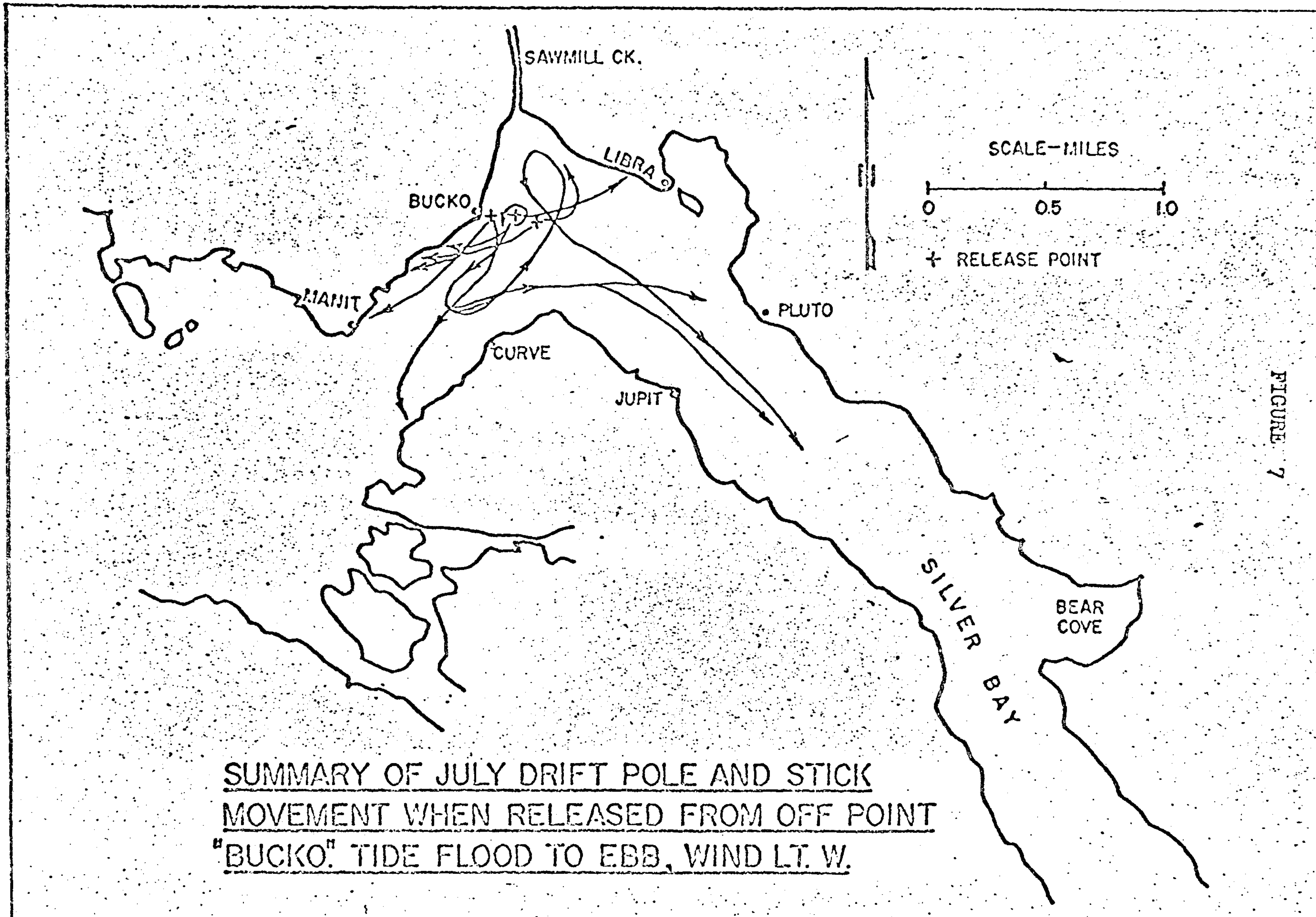
SUMMARY OF JULY DRIFT POLE AND STICK MOVEMENT WHEN RELEASED FROM EAST AND SOUTH OF SAWMILL COVE. TIDE FLOOD TO EBB. WIND LT. W.

FIGURE 6

figure (5) one can see that the river run-off is large enough to bring the drift poles into the main-channel. In further evidence of lateral flow, a quote from Alaska Water Pollution Control Board study is relevant; "Movement of the poles was towards the beach near Bucko or the beach near Libra". Looking at figure (6) this indicates that the motion toward Bucko would have been for an outgoing tide while ingoing towards Libra. This suggests a preferred passage, which is a result of both the coriolis force and topography of the inlet. Figure (7) further amplifies the fact that curvature affects the flow, giving a resultant lateral non-zero velocity.

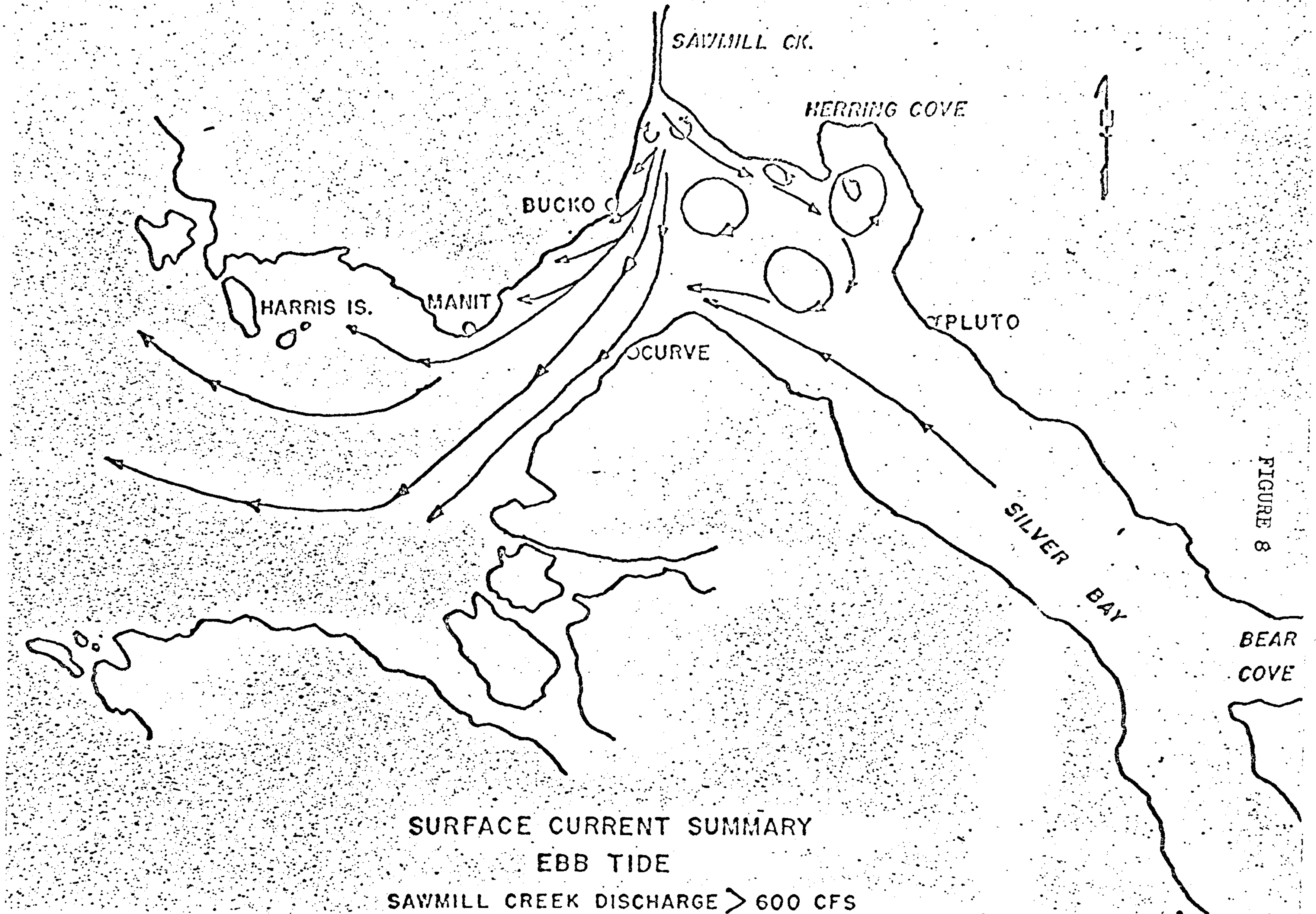
4.3 Effects of Tidal Motion.

By investigating current motion both at ebb and at flood tide, the effect of the tidal motion over a tidal period can be inferred. Figures (8) and (9) taken from the Alaska Water Pollution Control Board study give the surface current conditions at the ebb and flood tide. These figures suggest that resultant lateral velocities exist. Also, on further analysis it can be seen that the tidal actions on the water between Bucko and Manit are not equal and opposite for the ebb and flood, as has been suggested by the Silver Bay authors. The theory, as was mentioned in chapter 2, suggests that the



SUMMARY OF JULY DRIFT POLE AND STICK MOVEMENT WHEN RELEASED FROM OFF POINT "BUCKO." TIDE FLOOD TO EBB, WIND LT. W.

FIGURE 7



SURFACE CURRENT SUMMARY
EBB TIDE
SAWMILL CREEK DISCHARGE > 600 CFS

FIGURE 8

tidal velocity does not have equal and opposite effects in this section of the inlet.

4.4 Contributions of the River Run-off.

An important factor which has been neglected in the Silver Bay study is the effect of the river run-off. With large run-off, the tidal force effect would be dampened on the in-going tide. The amount of damping might be critical enough to cause strong vertical mixing, which in turn could cause the perturbations to become important in the analysis. In March there was very little river discharge in comparison to the July conditions. As a consequence the lateral flow should be more pronounced. This is evident by comparing figures (10) and (11) to figures (8) and (9).

4.5 Analysis of the Results of the Silver Bay study.

McAlister, Rattray, and Barnes did not obtain the result that their theory predicted. The velocity profile did not agree with the prediction. The salinity profile for the March data was in disagreement with their theory. The surface velocity for March was 120% of the critical velocity. These disagreements can in part be explained by the acceptance of invalid assumptions. In the light of the data presented by the Alaska Water Pollution Control Board, it is

Page 95 Missing

seen that the geometry of an inlet, must be considered as a major restraint to the flow. Even though the part of an inlet under survey has ideal boundary conditions, parts of the inlet which are not immediately adjacent to it can still affect the flow.

In the analysis of the stress field the stresses were computed in two different ways. The first way is by integrating the $\frac{\partial}{\partial z}(v_x' v_z')$ which are given in the Silver Bay study table (4) and (5). The alternate way is to compute the surface stresses from the mean wind data. The integrated values agree with the values obtained from the use of the mean wind.

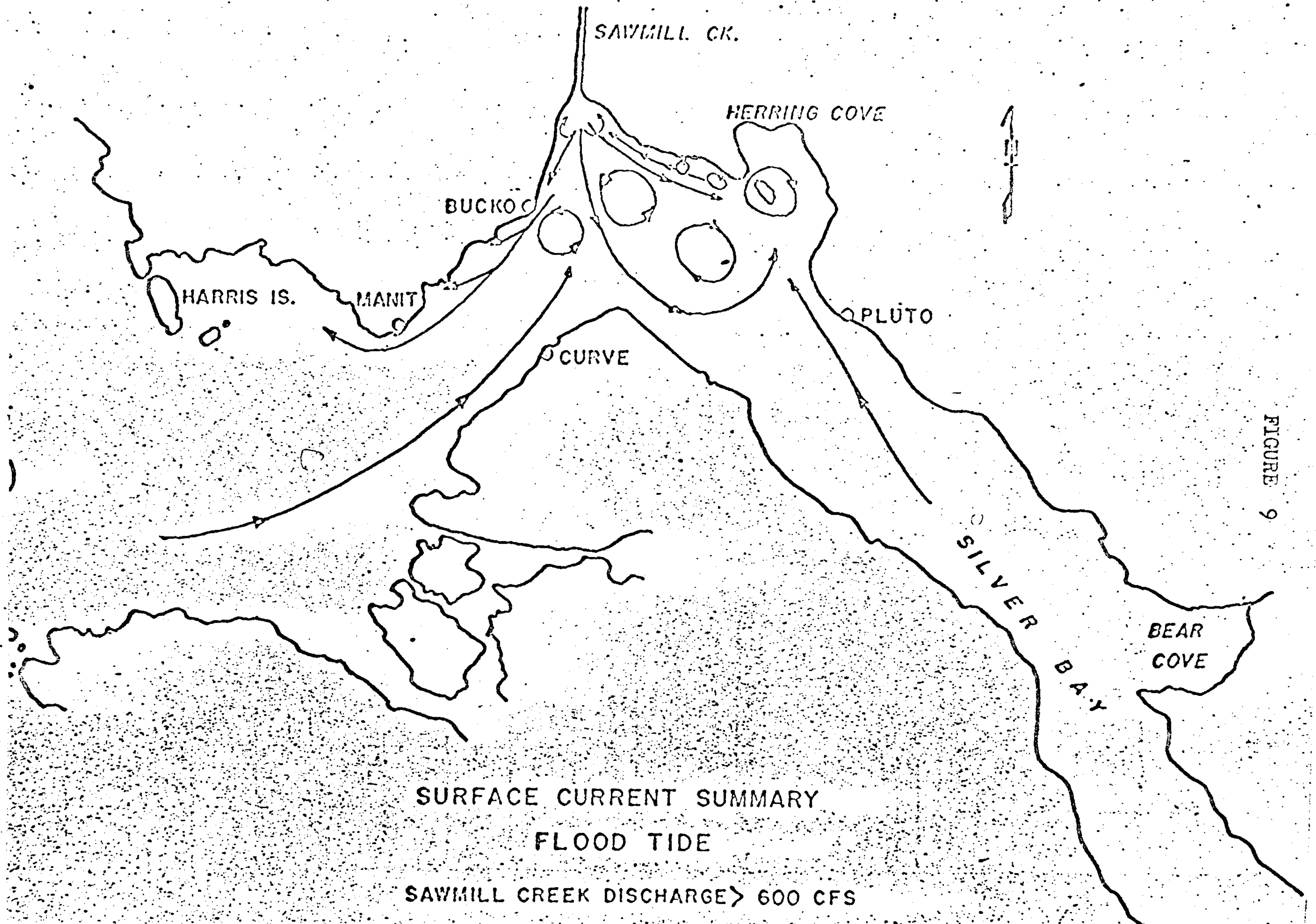
In regards to the pressure gradient, stress gradient, and inertial fields, the listing in the Silver Bay study table (4) is wrong. This has already been shown in Chapter 3. The questions that are still as yet unanswered, are those which determine the source of error. The possibilities of being

(i) Tidal motion

(ii) Lateral velicity component

(iii) Vertical component of coriolis $(-2\omega \cos\phi \, \bar{v}_z)$

or (iv) River run-off

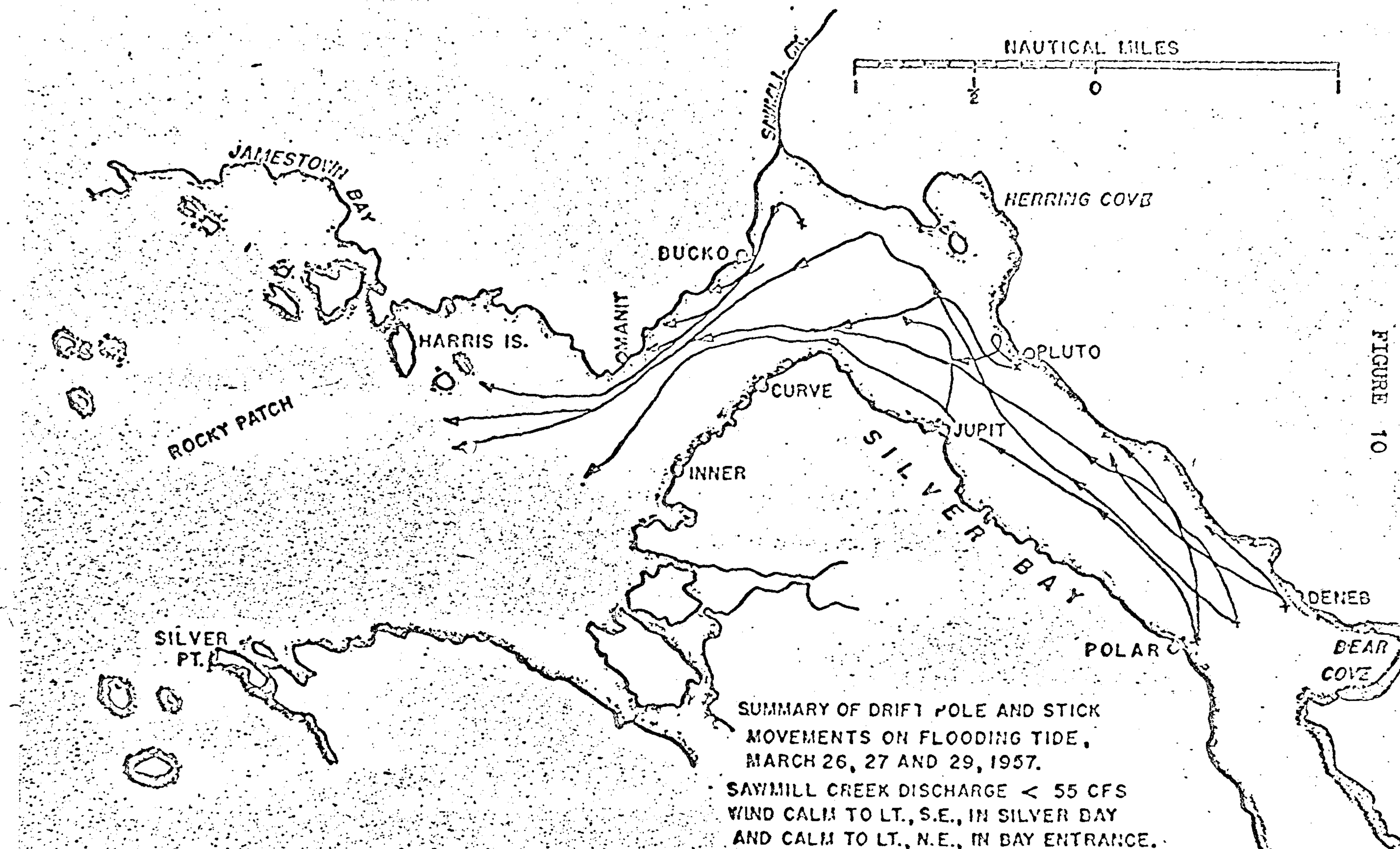


SURFACE CURRENT SUMMARY

FLOOD TIDE

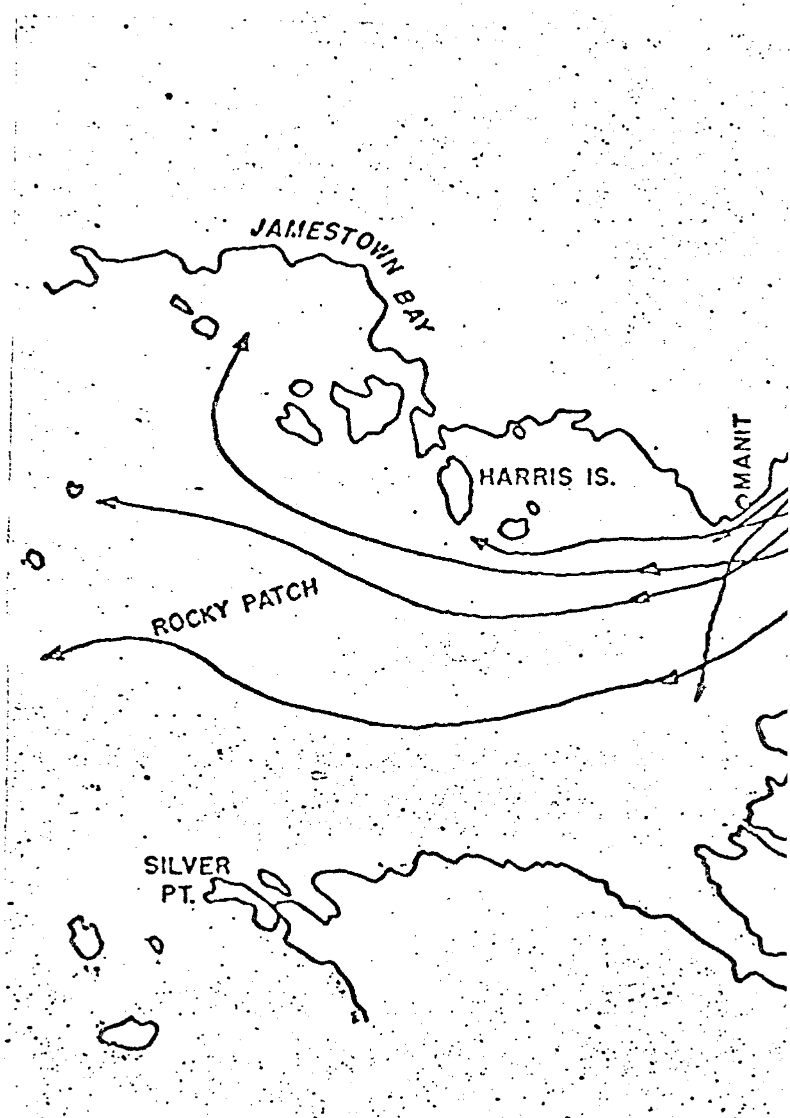
SAWMILL CREEK DISCHARGE > 600 CFS

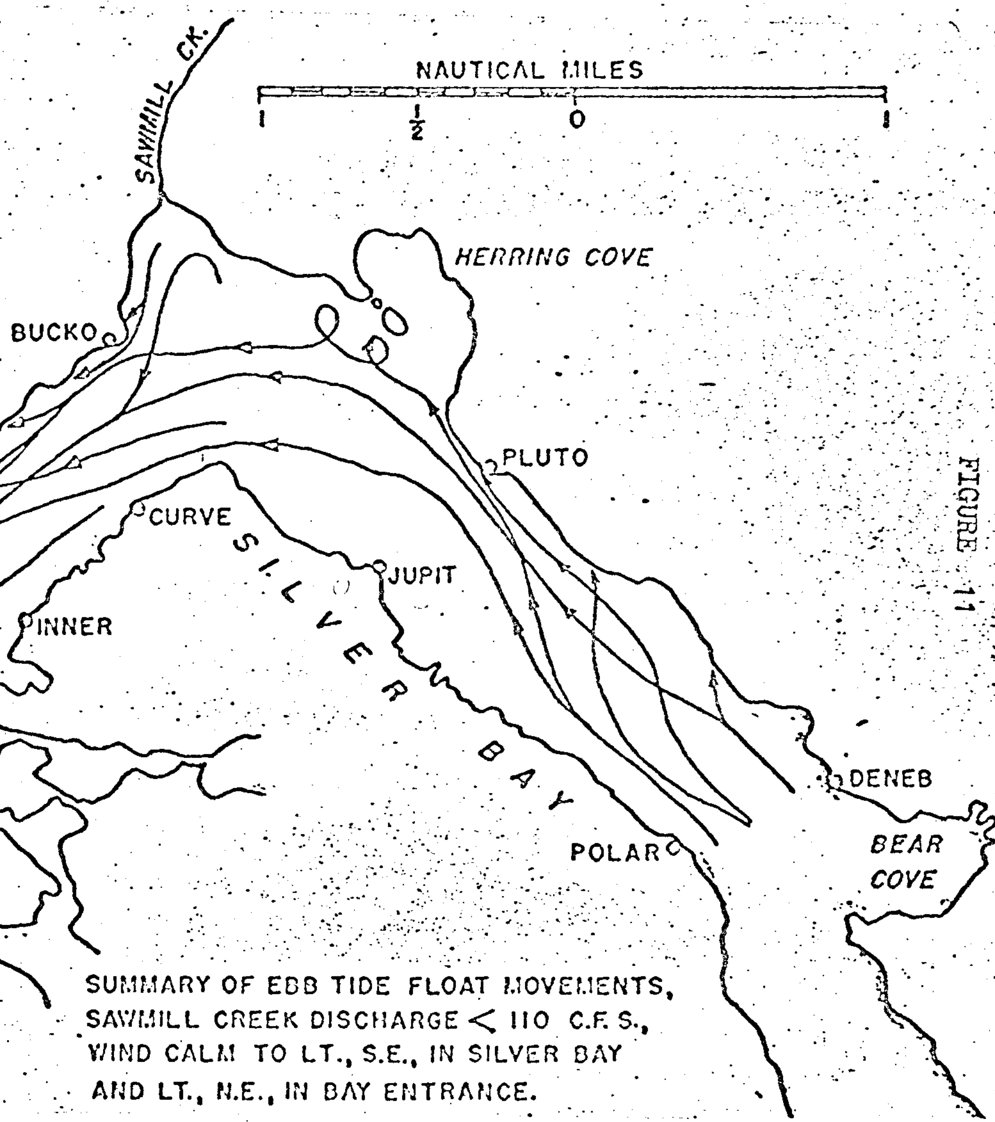
FIGURE 9



SUMMARY OF DRIFT POLE AND STICK MOVEMENTS ON FLOODING TIDE, MARCH 26, 27 AND 29, 1957.
SAWMILL CREEK DISCHARGE < 55 CFS
WIND CALM TO LT., S.E., IN SILVER BAY AND CALM TO LT., N.E., IN BAY ENTRANCE.

FIGURE 10


JAMESTOWN BAY
HARRIS IS.
ROCKY PATCH
SILVER PT.



SUMMARY OF EBB TIDE FLOAT MOVEMENTS,
SAWMILL CREEK DISCHARGE < 110 C.F.S.,
WIND CALM TO LT., S.E., IN SILVER BAY
AND LT., N.E., IN BAY ENTRANCE.

FIGURE 11

has been discussed, and the neglect of each may contribute to the over all error.

4.6 Summary of the Silver Bay Experiment.

The Silver Bay study was a first of its kind. The Navier-Stokes equation was numerically evaluated under certain assumptions. The results obtained did not satisfy the simplified equations of motion. The major assumptions were that $\bar{v}_y = 0$, and $\langle U_T \rangle = 0$ and the major omissions were to neglect to consider the curvature of the inlet, the term $-2\omega \cos\phi \bar{v}_z$ from the longitudinal equation of motion, and the amount of river run-off. It is found that it is important to consider the right form for the averaging operator. The assumptions $\bar{v}_y = 0$ and $\langle U_T \rangle = 0$ are not valid and the omissions produce errors. It is further found that a confidence level should be associated with the data. This confidence level would allow one to interpret the data, and at the same time the variance would determine how much the results could be trusted.

What one could now do is to repeat the whole Silver Bay experiment, but this time collect all the data that has been suggested. Further sophisticated analysis can be applied and this will be discussed in the next chapter.

CHAPTER 5

RECOMMENDATION FOR FURTHER STUDY

5.0 Introduction.

Physical conditions are always changing, and it is usually impossible to repeat an experiment under the exact same conditions. When considering the physical properties of an estuarine system, the boundary conditions are always changing. The data will be coupled with a time factor, and no static theory is adequate. The description of how the physical conditions vary with time is essential to understanding the physical phenomena present. The manner in which data is analyzed becomes very important in obtaining meaningful results. Further discussion of this will be given in this chapter.

5.1 General Considerations of the Fluid Flow.

In an incompressible fluid flow, the surface motion is coupled to the flow beneath in a very definite way. Studies of drifting objects deal with the translational motion, and it is quite clear that more insight can be gained by observing the mutual relative movement of fields of drifters in order to study the plane strain rates of flow. From this study, one might identify regions of up-welling, subsidence, vorticity and shear with all that may be inferred from these. Two points should be stressed in regards to taking of data.

(i) The water is a material continuum and cannot be properly traced by a field of two or three buoys. The buoy motion is only partially representative of the water flow and is not in any sense itself continuous. A finite field of buoys has many mathematical characteristics of a molecular array, in the sense that the buoys can, to a large extent, move independently of one another, and that they represent mass points seperated by large empty spaces. It is necessary to keep this in mind when working with a fixed array of a finite field of sensors. The separation between neighbors fixes the scale of turbulent phenomena that one can portray in this manner.

(ii) The other point to be made is that a single run of an experiment in a turbulent field does not yield much information about such a random process. It is in all essentials unrepeatable. Physical experiments must, therefore, be repeated until a statistically useful sample has been collected. Given that practical experiments must be accomplished in finite time with data from a finite array of points, one may ask how best to optimise the distribution of the data acquisition in time and space in order to assure the most representative and meaningful sample.

To sum up, the basic data desired by oceanographers are those needed to define water masses, differing as to temperature, salinity, colour and other characteristics, and to measure their modifications by turbulent diffusion, and by external energy flux from the air, the sky and the rotating earth. Since the ocean is a three dimensional continuum with only its upper surface exposed to the overview, observations must be taken with a view to kinematical derivation of the flow that is hidden from view. Except in a few special cases, remote sensing in a pure sense has no special merit other than possible convenience and economy. It will probably find its greatest use as a supplement to data acquired point by point in the ocean proper.

5.2 <u>General Discussion of the Numerical Method</u>.

In estuarine dynamics, data are collected so as to solve numerically the Navier - Stokes equation. This equation is

$$\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\nabla\underline{v} = -\frac{1}{\rho}\nabla p - g\nabla\chi + \frac{1}{3}\mu\nabla(\nabla\cdot\underline{v}) + \mu\nabla^2\underline{v} - 2\underline{\Omega}\times\underline{v}$$

and is subject to the equation of continuity

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\underline{v}) = 0$$

and the boundary conditions, which depend on the physical properties of the particular inlet.

There are many sources of error when collecting data, and so a statistical analysis of the data is needed. The analysis can be carried out using the theory of estimators. An estimator is a statistic, that is, a function of the sample values, which will provide numerical estimates of a parameter. Thus it is important to consider the desirable properties of these estimators. Fraser (1960) discusses the estimators, under the properties of being

(i) unbiased

(ii) consistent

(iii) sufficient.

These properties will be discussed in the next section.

The physical operations involved, in treating the data need to be well understood. One of these operations is the averaging process. The averaging operator is

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \{ \quad \} \, dt$$

and for periodic functions the operator reduces to

$$\frac{1}{nT} \int_0^{nT} \{ \quad \} \, dt$$

where T is the period of the periodic motion, and n is a positive integer. It is extremely important, that no functions are assumed to be periodic just because they might appear to be so. As an example of this, the tidal oscillation might be thought to be periodic. In general, the tidal velocity can be represented by a periodic function. In the case of a rectangular inlet, the velocity takes the form

$$U = U_0 \cos \phi$$

However, as has been shown, the physical properties of the inlet may be such that at ebb tide the velocity is

$$U = U_0 \cos \phi_0$$

and at flood tide

$$U = U_1 \cos \phi_1$$

where U_0, U_1, ϕ_0 and ϕ_1 can be functions of time and space. On assuming that the ebb tide and the flood tide have the same velocity function, terms like $\bar{v}_x \frac{\partial \bar{U}}{\partial x}$ and $\bar{U} \frac{\partial \bar{v}_x}{\partial x}$ would be considered as zero. But in actual fact they are $\bar{v}_x \frac{\partial}{\partial x}(U_0 - U_1)$ and $(U_0 - U_1) \frac{\partial \bar{v}_x}{\partial x}$, and the error introduced by assuming them to be zero might be considerable.

As a suggestion, tide gauges could be used, and the form of the tidal motion, then could be calculated. It would then be possible to determine the value of $|U_0 - U_1|$. A criterion could then be set up, such that if

$$|U_0 - U_1| \leq \epsilon \qquad \text{(depending on the inlet)}$$

then for all practical purposes, the tidal motion could be considered as periodic.

Since most inlets are not rectangular, it is strongly suggested that the lateral and vertical tidal components be considered. As has been seen in the work by Stewart (1957), the curvature of the James River was extreme enough for the lateral component of the tidal velocity to become important. Again, one should not assume the components to be periodic.

5.3 The Data Analysis.

For a numerical study of the Navier - Stokes equation the following data are required:

(i) Temperature (T)

(ii) Salinity (S)

(iii) Velocity Field ($\underline{V}$)

The T, S, and $\underline{V}$ will be functions of position (x,y,z) and time (t).

The temperature and salinity are usually measured using a Nansen bottle (see von Arx 1964). New electronic equipment has been developed, which makes use of a conductivity bridge to measure salinity. The velocity field can be measured by the use of current meters. The error in the measurement of temperature is of the order of 0.01° C. The salinity can be measured to 99.9 % accuracy.

5.31 Statistical Theory.

It has been stated that the estimators need to be unbiased, consistent and sufficient. For the estimator to be unbiased, the expectation value of the estimator must be the true mean of the statistic. As an example of this consider the following problem. Assume the need to measure $\mu = E\{X\}$ of a distribution, where $E\{X\}$ is the expectation of X. Then the estimator used to measure μ can be $\overline{X}$. Where

$$\overline{X} = \frac{1}{n} \sum_{i=1}^{n} X_i \; .$$

Then X is unbiased if $E\{\overline{X}\} = E\{X\}$.

Now

$$E\{\overline{X}\} = E\left\{\frac{X_1 + X_2 + \cdots + X_n}{n}\right\} = \frac{1}{n} E\{X_1 + X_2 + \cdots + X_n\}$$

$$= \frac{1}{n} \sum_{i=1}^{n} E\{X_i\} = E\{X\} = \mu$$

so $\bar{X}$ is an unbiased estimator of μ. An estimator τ is a consistent estimator of θ if the probability distribution of τ concentrates on the parameter value as the number of elements of data become large, i.e.

$$\Pr_{n \to \infty}\{\theta - \delta \leq \tau \leq \theta + \delta\} \to 1$$

where $\delta > 0$ and n is the amount of data. An estimator τ is a sufficient statistic if, given the value of τ, the conditional distribution is independent of the parameters. A statistic which is useful has all the three properties. Thus it is important that only estimators which have these properties be considered.

5.32 Hypothesis Testing.

Once the data has been collected, and the correct estimator is being used, a criterion needs to be developed so that a statistic can either be accepted or rejected. Roden (1968) examined some oceanographic data using suitable estimators. But since he did not apply a hypothesis test on his results, one is left to do further calculations before the results can be accepted.

As a first simple testing model consider a set of observations, relating a function f to depth. Consider the data to be taken over various days at positions (x_o, y_o, z) where (x_o, y_o) are fixed. Then a suitable estimator for the mean of f is

$$\bar{f}(x_o, y_o, z) = \frac{1}{n} \sum_{i=1}^{n} f_i(x_o, y_o, z)$$

where n is the number of observations. One can now consider the function

$$\sigma^2(f) = \frac{1}{n} \sum_{i=1}^{n} \left[f_i(x_o, y_o, z) - \bar{f}(x_o, y_o, z) \right]^2$$

where $\sigma^2(f)$ is the variance of f. Then the value of $\bar{f}(x_o, y_o, z)$ can either be accepted or rejected depending on whether $\sigma^2(f) \leq \sigma_o^2(f)$ or not, where $\sigma_o^2(f)$ is the maximum variance tolerable for the calculations. This means if $\sigma^2(f) > \sigma_o^2(f)$, the data does not have a required accuracy. The maximum variance has to be calculated. This can be done by considering statistical distributions. Fraser (1960, pg. 374) summarizes the method.

The statistical analysis of the data, is an integral part of a complete analysis of an experiment. One can have confidence in the results, and the amount of error is known.

Also when repeating an experiment, by knowing where the errors are introduced more care can be taken, even new techniques to obtain better results could be devised.

5.4 The Equations of the Analysis.

A summary of all the equations needed for the analysis will now be given. The main equations are the component equations of motion for the general mean flow. These are given as equations (32), (33) and (34). When allowing for some legitimate assumptions, the equations become those of (40), (41) and (42). The energy equation for the mean motion, equation (46), is also an important equation. From the energy equation, useful information about the energy distribution can be found. The time average operator, equation (36), is important in the analysis, since the mean equations are the ones being investigated. As for the equations involved in the numerical calculations, they are not individually important. Chapter three, which discusses the numerical study, can be considered as a chapter of algebraic manipulations.

5.5 Modelling of an Estuary.

The work that has been discussed in all the previous sections, has been in reference to testing the equation of flow. The Silver Bay study was an effort to investigate the dynamics of the flow, and basically this was the testing of the Navier - Stokes equation under simplifying assumptions. The attempt by the Silver Bay authors was the first of its kind. Understandably some mistakes in their analysis has been found. There was also a lack of sufficient data.

Learning from the Silver Bay study one can now proceed to the next step of sophistication. More reliable data is necessary. A statistical study should be employed, to determine the usefulness of the data. The consequences of all the assumptions need to be investigated. Once there is consistency between the theory and the experimental values, an analytical study could be made.

A possible analytical approach, would be to obtain analytical representation for the salinity and temperature distribution. The salinity, for example, could be expressed in terms of some geometric parameters. A possible set of parameters are, (i) distance from the point of measurement to the bottom of the estuary, (ii) distance from the side, and (iii) distance from the head of the inlet.

As an example of these possibilities, some salinity data was examined. This salinity data was collected by Rosenberg et. al. (1967) in Cook Inlet, S. E. Alaska. Carrying out a simple analysis for the surface data, it is found that the salinity can be represented by

$$S = \alpha \log(X + \beta)$$

where α, β are constants, and X is the distance from the fresh water source. This type of mathematical form for the salinity, would enable a comparison to be made between various inlets.

An analogy can be made between meteorological and oceanographic data. In meteorology, analytical expressions for transport coefficients have been calculated. Stability criteria have been evaluated for the flow of air masses. If enough data were available similar calculations should be possible in a water regime.

5.6 Recommendation for Further Study.

Physical oceanography is still a relatively new science, and the study of estuarine dynamics has not produced any sophisticated new theory. What is most lacking is data. Much more data needs to be collected. The type of data required has been discussed, and it is essential that it be obtained to produce a useful theory. Meteorological data should be collected simultaneously with the oceanographic data. Radiation measurements should be made, so that heat budget studies can be carried out. Winds values will assist in calculating surface stresses. Geological studies of bottom properties will enable one to determine the laws applicable to bottom stresses. All in all, one should take all the data possible even though it may not be immediately useful. Later a quantity that was not measured may be important enough, that the whole experiment might need to be repeated. For example no $\bar{v}_y$ data are available from the Silver Bay study. Assuming that at the time $\bar{v}_y$ could have been neglected, if some data had been taken, a re-calculation of the equations could be made without repeating the whole experiment.

One can summarize the recommendations in a list form. If physically possible

(i) take data of all (T, S, $\underline{V}$)

and at the same time

(ii) take the associated meteorological data

then

(iii) a statistical analysis must be carried out on the data to check its validity, and further an

(iv) analytical representation of the oceanographic parameters should be one of the principle aims, so as to be able to characterize inlets. As a concluding remark one hopes that in the advent of new data becoming available, a more consistent theory can be developed.

Bibliography

Alaska Water Pollution Control Board, 1957

Silver Bay Water Pollution Control Studies Near Sitka, Alaska; Report No. 10

Anonymous, 1967

Accuracy of Current Meter Readings

Hydro Products Pub., p. 117

Bowden, K. F., Fairbairn, L. A., and Hughes, p., 1959

The Distribution of Shearing Stresses in a Tidal Current

Geophysical Journal Roy. Astron. Soc. Lond. 2 4 p. 288 - 305

Bowden, K. F., 1962

Measurements of Turbulence near the Sea Bed in a Tidal Current

J. Geophysical Research, 67 8 p. 3181 - 3186

Bowden, K. F. and Howe, M. R., 1963

Observations of Turbulence in a Tidal Current

J. Fluid Mech. 17 p. 271 - 284

Cameron, W. M., 1951

On the Dynamics of Inlet Circulation

Doctoral Dissertation, Scripps Inst. of Ocean., Univ. of California

Chandrasekhar, S., 1961

Hydrodynamic and Hydromagnetic Stability

Oxford Press, Oxford, England

Defant, A., 1961

Physical Oceanography

Pergamon Press, New York, Vol. 2, p. 293

Dronkers, J. J., 1964

Tidal Computations

Interscience Publishers, New York, p. 518

Fraser, D. A. S., 1960

Statistics, an Introduction

John Wiley and Sons, New York, p. 214 - 243

Fraser D. A. S., 1960

Statistics, an Introduction

John Wiley and Sons, New York, p. 374

Grant, H. L., Stewart, R. W. and Moilliet A., 1962

Turbulence Spectra from a Tidal Channel

J. Fluid Mech. 12 2 p. 241 - 268

Hansen, D. V. and Rattray, M. Jr., 1966

New Dimensions in Estuary Classification

Limnol. and Ocean. XI 3 p. 319 - 326

Hansen, D. V. and Rattray, M. Jr., 1966

New Dimensions in Estuary Classification

Limnol. and Ocean. XI 3 p. 319 - 326

Harleman, D. R. F. and Ippen, A. T., 1961

The Turbulent Diffusion and Convection of Saline Water in an Idealized Estuary

Pub. No. 51

Kinsman, B., 1966

Waves and Tides

Special Lecture Course for John Hopkins Univ.

Unpublished, p. 94, p. 143

Landau, L. D. and Leftshitz, E. M., 1951

Fluid Mechanics

Pergamon Press, New York, p. 53

Lamb, Sir Horace, 1932

Hydrodynamics

Cambridge Univ. press, New York, 6th Edition

Lamb, Sir Horace, 1932

Hydrodynamics, Cambridge Univsity Press, New York, sixth Edition, p. 250 - 341

Lamb, Sir Horace, 1939

Hydrodynamics

Cambridge University Press, New York Edition, sixth Edition, p. 262

Lesser, R. M., 1951

Some Observations of the Velocity Profile Near the Sea Floor

Trans, Amer. G. Geophys. Un. 32 2 p. 207 - 211

Maximan, L. C. and Brown, G. W., 1955

A Theory of Tidal Mixing in a Vertically Homgeneous Estuary

J. Mar. Res. 14 2 p. 157 - 175

McAllister, W. B., Rattray, M. and Barnes, C. A., 1959

The Dynamics of a Fiord Estuary; Silver Bay Alaska

Tech. Rep. No. 62, Office of Naval Research Contract Nonr - 477(10) and Nonr - 477(01)

Munn, R. E., 1966

Descriptive Micro - Metereology

Academic Press, p. 57, p. 167

Neumann, G. and Pierson, W. J. Jr., 1966

Principles of Physical Oceanography

Prentice - Hall, p. 123, p. 212

Pickard, G., 1953

Oceanography of British Columbia Mainland Inlets 1 Water Characteristics

Progress Reports of the Pacific Coast Station, Fisheries Research Board Canada, No. 96, p. 3 - 6, Oct. 1963

Pickard, G., 1961

Oceanographic Features of Inlets in the British Columbia Mainland Coast

J. Fish. Res. Bd. Canada, 18 6 p. 907 - 999

Pritchard, D. W., 1954

Factors Affecting the Disposal of Fission Products in Estuaries and Inshore Environments

Second U. N. Geneva Conference, Pergamon Press London

Pritchard, D. W., 1956

The Dynamic Structure of a Coastal Plain Estuary

J. Mar. Res. 15 1 p. 33 - 42

Pritchard, D. W. and Kent, R. E., 1956

A Method for Determining Mean Longitudinal Velocities in a Coastal Plain Estuary

J. Mar. Res. 15 1 p. 81 - 91

Proudman, J., 1953

Dynamical Oceanography

Dover Publication Inc., New York, 409 pps.

Rattray, M., Jr. and Hansen, D. V., 1962

A Similarity Solution for Circulation in an Estuary

J. Mar. Res. 20 2 p. 121 - 133

Rattray, M., Jr., and Hansen, D. V., 1965

Gravitational Circulation in Straits and Estuaries

J. Mar. Res. 23 2 p. 104 - 122

Roden G. I., 1968

Spectral Analysis and Interpretation of Salinity - Temperature Depth Records

J. Geophys. Res. 73 2 p. 535 - 539

Rosenberg, D. H., Burrell, D. C., Natarajon, K. V. and Hood, D. W., 1967
Oceanography of Cook Inlet. Special Report Univ. of Alaska - Institute of Marine Science, Report No.67-3

Rutherford, D. E., 1959
Fluid Mechanics
Oliver and Boyd, New York, p. 194

Schmitz, H. P., 1962
A Relation Between the Vectors of Stress, Wind and Current at Water Surfaces and Between the Shearing Stress and Velocities at Solid Boundaries
Deut. Hydrog. Zeit. 15 1 p. 23 - 36

Sokolnikoff, I. S., and Redheffer, R. M., 1958
Mathematics for Physics and Engineering
McGraw - Hill, p. 362 - 370

Sokolnikoff, I. S. and Redheffer, R. M., 1966
Mathematics for Physics and Engineering
McGraw - Hill, p. 347

Stewart, R. W., 1957
A Note on the Dynamic Balance in Estuarine Circulation
J. Mar. Res. 16 1 p. 34 - 38

Stokes, G. G., 1847

On the Theory of Oscillating Waves

Trans. Camb. Phil. Soc., 8 p. 441

Stommel, H., 1951

Recent Developments in the Study of Tidal Estuaries

J. Mar. Res., 13 1 p. 133 - 144

Stommel, H. and Farmer, H. G. 1952

On the Nature of Estuarine Circulation

Tech. Rep., June 1952, Office of Naval Res. Contract

Nonr 27701 (NR 083-004)

Tully, J. P., 1958

On the Structure, Entrainment, and Transport in

Estuarine Embayments

J. Mar. Res. 17, p. 523 - 535

Von Arx W., 1964

An Introduction to Physical Oceanography

Addison and Wesley, New York, p. 244 - 284, p. 255 - 258